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Solving the local cohomology problem in U(1) chiral gauge theories within a finite lattice

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Abstract: In the gauge-invariant construction of abelian chiral gauge theories on the lattice based on the Ginsparg-Wilson relation, the gauge anomaly is topological and its cohomologically trivial part plays the role of the local counter term. We give a prescription to solve the local cohomology problem within a finite lattice by reformulating the Poincaré lemma so that it holds true on the finite lattice up to exponentially small corrections. We then argue that the path-integral measure of Weyl fermions can be constructed directly from the quantities defined on the finite lattice.

Keywords: Lattice Gauge Field Theories, Gauge Symmetry, Anomalies in Field and String Theories.

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Contents

1. Introduction

Recently it turned out that lattice gauge theory can provide a framework for non-perturbative study of chiral gauge theories, despite the well-known problem of the species doubling. The clue to this development is the construction of gauge-covariant and local lattice Dirac operators satisfying the Ginsparg-Wilson relation [1]–[6],

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D \,. \tag{1.1}$$

An explicit solution of the Ginsparg-Wilson relation was derived from the overlap formalism based on domain wall fermion and is referred as the overlap Dirac operator.[1] It has made

[1]The overlap formalism proposed by Narayanan and Neuberger [7]–[16] gives a well-defined partition function of Weyl fermions on the lattice, which nicely reproduces the fermion zero mode and the fermion-number violating observables ('t Hooft vertices) [17, 18, 19]. Through the recent re-discovery of the Ginsparg-Wilson relation, the meaning of the overlap formula, especially the locality properties, become

possible to realize exact chiral symmetry on the lattice [28] and also opened a route to the gauge invariant construction of anomaly-free chiral gauge theories on the lattice [29]–[33].

In the gauge-invariant construction of chiral gauge theories on the lattice, one of the crucial steps is to establish the exact cancellation of gauge anomaly at a finite lattice spacing. In abelian chiral gauge theories [30, 34],[2] it has been achieved through the cohomological classification of the chiral anomaly [29, 36, 37], which is given in terms of lattice Dirac operator satisfying the Ginsparg-Wilson relation [28]–[42],

$$q(x) = \mathrm{tr}\left\{\gamma_5(1-aD)(x,x)\right\} \tag{1.2}$$

and is a topological field for the admissible lattice gauge fields satisfying the bound[3]

$$\| \, 1 - P_{\mu\nu}(x) \, \| < \epsilon \,, \qquad \epsilon < \frac{1}{30} \tag{1.3}$$

$$P_{\mu\nu}(x) \equiv U(x,\mu)U(x+\hat{\mu},\nu)U(x+\hat{\nu},\mu)^{-1}U(x,\nu)^{-1} \,. \tag{1.4}$$

For an anomaly-free multiplet of Weyl fermions satisfying the anomaly cancellation condition of the U(1) charges,

$$\sum_\alpha e_\alpha^3 = 0 \,, \tag{1.5}$$

it has been shown that the chiral anomaly is cohomologically trivial,

$$\sum_\alpha e_\alpha q^\alpha(x) = \partial_\mu^* k_\mu(x) \,, \qquad q^\alpha(x) = q(x)|_{U \to U^{e_\alpha}} \,, \tag{1.6}$$

where $k_\mu(x)$ is a certain gauge-invariant and local current. The cohomologically trivial part of the chiral anomaly is then used in the gauge-invariant construction of the Weyl fermion measure. In short, it plays the role of the local counter term in the effective action for the Weyl fermions.[4]

For the practical computation of observables in the lattice abelian chiral gauge theories, it is required to compute the Weyl fermion measure for every admissible configuration.

clear from the point of view of the path-integral. The gauge-invariant construction by Lüscher [30] based on the Ginsparg-Wilson relation provides a procedure to determine the phase of the overlap formula in a gauge-invariant manner for anomaly-free chiral gauge theories. For Dirac fermions, the overlap formalism provides a gauge-covariant and local lattice Dirac operator satisfying the Ginsparg-Wilson relation [1, 2, 16, 4, 6]. The overlap formula was derived from the five-dimensional approach of domain wall fermion proposed by Kaplan [20]. In its vector-like formalism [21, 22, 23, 24], the local low energy effective action of the chiral mode precisely reproduces the overlap Dirac operator [25, 26, 27].

[2] See also [35] for a gauge-invariant construction of abelian chiral gauge theories in non-compact formulation.

[3] It has been shown by Neuberger [43] that the constant 1/30 in the above bounds can be improved to $1/6(2+\sqrt{2})$.

[4] For nonabelian chiral gauge theories, the local cohomology problem can be formulated with the topological field in 4+2 dimensional space. [44, 31, 32, 33, 45] So far, the exact cancellation of gauge anomaly has been shown in all orders of the perturbation expansion for generic nonabelian theories [46, 47, 48], and non-perturbatively for $\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ electroweak theory, both in the infinite lattice [49]. In the five-dimensional approach using the domain wall fermion [20]–[27], the local cohomology problem can be formulated in 5+1 dimensional space [50].

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However it seems difficult to follow the steps given in [30] literally. The first problem is the use of the infinite lattice in order to make sure the locality property of the cohomologically trivial part.[5] As a closely related problem, the vector-potential-representative of the link variable used in the cohomological analysis is unbounded. The second problem is the use of the continuous interpolations in the space of the admissible U(1) gauge fields.

The purpose of this paper is to give a prescription to solve the local cohomology problem *within a finite lattice* which applies directly to the chiral anomaly on the finite lattice,

$$q(x) = \mathrm{tr}\left\{\gamma_5(1 - D_L)(x,x)\right\}, \tag{1.7}$$

where $D_L(x,y)$ is the finite-volume kernel of the Dirac operator. With this method, we will show that the current $k_\mu(x)$, which gives the the cohomologically trivial part of the above chiral anomaly, can be obtained directly from the quantities calculable on the finite lattice. Then we will argue that the measure of the Weyl fermions can be also constructed directly from the quantities defined on the finite lattice.[6] For our purpose, we first examine the Poincaré lemma, which is originally formulated on the infinite lattice [29]. We will show that the lemma can be reformulated so that it holds true on the finite lattice up to exponentially small corrections of order $O(\mathrm{e}^{-L/2\varrho})$, where L is the lattice size and ϱ is the localization range of the differential forms in consideration.[7] (Section 4)

We next examine the one-to-one correspondence between the link variable and the vector potential with a certain good locality property, which is originally derived in the infinite lattice [29]. We will show that a similar one-to-one correspondence with the desired locality property can be formulated for the admissible U(1) gauge fields on the finite lattice, by separating the link variables into the part which is responsible to the magnetic flux (the constant mode of the field strength) and the part of the local and dynamical degrees of freedom around the magnetic flux. (Section 5)

Equipped with the modified Poincaré lemma and the vector potentials for the admissible U(1) gauge fields on the finite lattice, we will perform the cohomological analysis of the chiral anomaly directly on the finite lattice. Through this analysis, we will derive a formula by which the cohomologically trivial part of the chiral anomaly is given in terms of the quantities defined on the finite lattice. (Section 6)

In this paper, we will focus on how to obtain the cohomologically trivial part (the local counter term) directly on the finite lattice. We do not claim that the cohomological classification of the chiral anomaly can be completed within the setup of the finite lattice.

[5] The cohomologically trivial part is, so far, constructed in two steps: the local cohomology problem is first solved in the infinite lattice [32] and then the corrections required in the finite lattice are constructed and added [52]. Since the lattice Dirac operator satisfying the Ginsparg-Wilson relation should have the exponentially decaying tail [53, 54], the local fields in consideration should have the infinite number of components. Moreover, the vector potentials used in this analysis are not bounded.

[6] The continuous interpolation in the space of the admissible gauge fields is also a crucial technique in the above construction. We will discuss the issue how to implement the continuous interpolation numerically in the forthcoming paper.

[7] For the ultra-local tensor fields on a finite lattice, the Poincaré lemma has been formulated by Fujiwara et al in [55].

Rather we will use some of the results in the infinite volume as inputs in order to establish the exact cancellation of gauge anomaly on the finite lattice, as we will see in section 6. For the presentation of our result, we follow closely the convention and the notation adopted in [29], so that the necessary modification of the cohomological analysis in the finite lattice becomes clear.

2. U(1) gauge fields on a finite-volume lattice

We consider a finite four-dimensional lattice of size L with periodic boundary conditions and choose lattice units. The U(1) gauge fields on such a lattice may be represented through periodic link fields,

$$U(x,\mu) \in \mathrm{U}(1)\,, \qquad x=(x_1,\ldots,x_4)\in \mathbb{Z}^4\,, \tag{2.1}$$

$$U(x+L\hat{\nu},\mu) = U(x,\mu) \qquad \text{for all } \mu,\nu = 1,\ldots,4\,, \tag{2.2}$$

on the infinite lattice. The independent degrees of freedom are then the link variables at the point x in the block

$$\Gamma_{4\,[x_0]} = \left\{ x\in\mathbb{Z}^4 \,\middle|\, -\frac{L}{2} \le x_\mu - x_{0\mu} < \frac{L}{2} \right\} \tag{2.3}$$

where x_0 is a certain reference point. L is assumed to be an even integer. Under gauge transformations

$$U(x,\mu) \to \Lambda(x) U(x,\mu) \Lambda(x+\hat{\mu})^{-1}\,, \tag{2.4}$$

the periodicity of the field will be preserved if $\Lambda(x)\in \mathrm{U}(1)$ is periodic.

We impose the so-called admissibility condition on the U(1) gauge fields:

$$|F_{\mu\nu}(x)| < \epsilon \qquad \text{for all } x,\mu,\nu\,, \tag{2.5}$$

where the field tensor $F_{\mu\nu}(x)$ is defined through

$$F_{\mu\nu}(x) = \frac{1}{i}\ln P_{\mu\nu}(x)\,, \qquad -\pi < F_{\mu\nu}(x) \le \pi\,. \tag{2.6}$$

We require this condition because it ensures that the overlap Dirac operator [2, 4] is a smooth and local function of the gauge field for $\epsilon < 1/30$ [6].

The admissible U(1) gauge fields on the finite lattice can be classified by the magnetic fluxes $m_{\mu\nu}$, where

$$m_{\mu\nu} = \frac{1}{2\pi}\sum_{s,t=0}^{L-1} F_{\mu\nu}(x+s\hat{\mu}+t\hat{\nu}) \tag{2.7}$$

(integers independent of x). The following field is periodic and can be shown to have constant field tensor equal to $2\pi m_{\mu\nu}/L^2 (<\epsilon)$:

$$V_{[m]}(x,\mu) = \mathrm{e}^{-\frac{2\pi i}{L^2}\left[L\delta_{\tilde{x}_\mu,L-1}\sum_{\nu>\mu} m_{\mu\nu}\tilde{x}_\nu + \sum_{\nu<\mu} m_{\mu\nu}\tilde{x}_\nu\right]}\,, \tag{2.8}$$

where the abbreviation $\tilde{x}_\mu = x_\mu \bmod L$ has been used. Then any admissible U(1) gauge field in the topological sector with the magnetic flux $m_{\mu\nu}$ may be expressed as

$$U(x,\mu) = \tilde{U}(x,\mu)\, V_{[m]}(x,\mu)\,. \tag{2.9}$$

We may regard $\tilde{U}(x,\mu)$ as the actual local and dynamical degrees of freedom in the given topological sector.

3. Local composite fields on the finite lattice

Under the admissibility condition, the space of the U(1) lattice gauge fields on the finite-volume lattice is separated into topological sectors labeled by the magnetic flux $m_{\mu\nu}$. Over the topological sectors, the gauge field can not be deformed continuously to each other without breaking the admissibility condition. Then a composite field of gauge field, which is a functional defined over the field space, could be defined independently in each topological sectors. Moreover, even in each topological sectors, the link variables are not independent each other and then the locality of composite fields of gauge field is not completely obvious. Therefore we need to clarify what exactly a local composite field on the finite-volume lattice means.

We first note the fact that the finite-volume kernel $D_L(x,y)$ of a lattice Dirac operator of a Ginsparg-Wilson type fermion may be represented by the kernel in the infinite lattice $D(x,y)$ which is restricted to the periodic link field:

$$D_L(x,y) = D(x,y) + \sum_{n\in\mathbb{Z}^4, n\neq 0} D(x, y+nL)\,, \tag{3.1}$$

$$D_L(x,y) = D(x,y) + \mathrm{O}(\mathrm{e}^{-L/\varrho})\,. \tag{3.2}$$

The second equation follows from the requirement of the locality for the lattice Dirac operator D in the infinite lattice [6, 29]. Namely, it is required that $D(x,y)$ is a sum of strictly local operators,

$$D(x,y) = \sum_{k=1}^{\infty} D_k(x,y)\,, \tag{3.3}$$

which are localized on the blocks with side-lengths $2k$. Moreover these kernels and their derivatives $D_k(x,y;y_1,\nu_1,\ldots,y_n,\nu_n)$ with respect to the gauge field variables $U(y_1,\nu_1)$, $\ldots$, $U(y_n,\nu_n)$ are required to satisfy the bounds

$$\|D_k(x,y;y_1,\nu_1,\ldots,y_n,\nu_n)\| \leq a_n k^{p_n} \mathrm{e}^{-\theta k} \qquad (n \geq 0) \tag{3.4}$$

where the constants $a_n, p_n \geq 0$ and $\theta > 0$ can be chosen to be independent of the gauge field. As a consequence we have

$$\|D(x,y;y_1,\nu_1,\ldots,y_n,\nu_n)\| \leq a'_n (1 + \|x-z\|^{p_n}) \mathrm{e}^{-\|x-z\|/\varrho}\,, \tag{3.5}$$

for a constant a'_n and the integer $p_n \geq 0$, where z is chosen from $y, y_1, \ldots, y_n$ so that $\|x-z\|$ is the maximum. The localization range is given by $\varrho = 2/\theta$. In the case of the overlap Dirac operator, it has been proved that these requirements are satisfied for all admissible gauge fields [6].

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From this observation, it is reasonable, for our purpose, to specify the notion of a local composite field on the finite-volume lattice as follows:

Locality property of local composite fields (I): *we refer a composite field $\phi(x)$ on the finite periodic lattice as local if $\phi(x)$ can be expressed as the sum of two parts, the local composite field $\phi_\infty(x)$ defined in the infinite lattice and restricted with a periodic gauge field and the finite-volume correction $\Delta\phi_\infty(x)$:*

$$\phi(x) = \phi_\infty(x) + \Delta\phi_\infty(x)\,; \qquad |\Delta\phi_\infty(x)| \leq c\,L^q\,\mathrm{e}^{-L/2\varrho} \tag{3.6}$$

for a constant c and an integer $p \geq 0$, where ϱ is the localization range of $\phi(x)$. $\phi_\infty(x)$ is required to be local in the sense that it can be written as a series

$$\phi_\infty(x) = \sum_{k=1}^{\infty} \phi_{k\infty}(x) \tag{3.7}$$

of strictly local fields $\phi_k(x)$ which are localized on the blocks with side-lengths proportional to k, and these fields and their derivatives $\phi_k(x; y_1, \nu_1, \ldots, y_m, \nu_m)$ with respect to the gauge field variables are bounded by

$$|\phi_{k\infty}(x; y_1, \nu_1, \ldots, y_m, \nu_m)| \leq c_m k^{q_m} \mathrm{e}^{-k/\varrho} \qquad (m \geq 1)\,, \tag{3.8}$$

where the constants $c_m, q_m \geq 0$ and the localization range $\varrho > 0$ are independent of the gauge field.

As to the locality of a composite field on the finite-volume lattice, it turns out to be convenient, for our purpose, to specify its property in more detail. We note again eq. (3.1), the relation of the finite-volume kernel $D_L(x,y)$ to that in the infinite lattice $D(x,y)$, and also the fact that the differentiation of $D_L(x,y)$ with respect to the periodic link field $U(z,\nu)$ is related to the differentiation of $D(x,y)$ with respect to the generic link field in the infinite lattice as follows: let $\tilde\eta(x,\nu)$ be a periodic variation of the link field, while $\eta(x,\nu)$ is an arbitrary variation in the infinite lattice and then we have

$$\begin{aligned}
\frac{\delta D_L(x,y)}{\delta\tilde\eta(z,\nu)} &= \frac{\delta D(x,y)}{\delta\tilde\eta(z,\nu)} + \sum_{n\in\mathbb{Z}^4, n\neq 0} \frac{\delta D(x,y+nL)}{\delta\tilde\eta(z,\nu)} \\
&= \frac{\delta D(x,y)}{\delta\eta(z,\nu)} + \sum_{m\in\mathbb{Z}^4, m\neq 0} \frac{\delta D(x,y)}{\delta\eta(z+mL,\nu)} + \sum_{n\in\mathbb{Z}^4, n\neq 0}\sum_{m\in\mathbb{Z}^4} \frac{\delta D(x,y+nL)}{\delta\eta(z+mL,\nu)}\,,
\end{aligned} \tag{3.9}$$

where, after the variation, parameters $\tilde\eta$ and η are set to be zero. From these relations we can see that $D_L(x,y)$ is also a sum of operators,

$$D_L(x,y) = \sum_{k=1}^{\infty} D_{Lk}(x,y), \tag{3.10}$$

$$D_{Lk}(x,y) = D_k(x,y) + \sum_{n\in\mathbb{Z}^4, n\neq 0} D_k(x,y+nL)\,, \tag{3.11}$$

where $D_{Lk}(x,y)$ is strictly local in the sense that the kernel itself and their derivatives $D_{Lk}(x,y;y_1,\nu_1,\ldots,y_n,\nu_n)$ with respect to the periodic gauge field variables $U(y_1,\nu_1)$, …, $U(y_n,\nu_n)$ vanishes identically if $2k < \max_{z=y,y_1,\ldots,y_n} \|x-z\|$:

$$D_{Lk}(x,y;y_1,\nu_1,\ldots,y_n,\nu_n) = 0 \qquad \text{if} \quad 2k < \max_{z=y,y_1,\ldots,y_n} \|x-z\| . \tag{3.12}$$

Moreover we can infer the following bounds:

$$\|D_{Lk}(x,y;y_1,\nu_1,\ldots,y_n,\nu_n)\| \leq b_n k^{p_n} \mathrm{e}^{-\theta k} \qquad (n \geq 0) , \tag{3.13}$$

where the constants $b_n, p_n \geq 0$ and $\theta > 0$ can be chosen to be independent of the gauge field and the lattice size L. As a consequence we have

$$\|D_L(x,y;y_1,\nu_1,\ldots,y_n,\nu_n)\| \leq b'_n (1+\|x-z\|^{p_n}) \mathrm{e}^{-\|x-z\|/\varrho} \qquad (n \geq 0), \tag{3.14}$$

for a constant b'_n and the integer $p_n \geq 0$, where z is chosen from $y, y_1, \ldots, y_n \in \Gamma_{4\,[x]}$ so that $\|x-z\|$ is the maximum.[8] Therefore, it is reasonable to require further the following property for a local composite field on the finite-volume lattice:

Locality property of local composite fields (II): *a local composite field $\phi(x)$ on the finite periodic lattice is also expressed as the sum of local fields,*

$$\phi(x) = \sum_{k=1}^{\infty} \phi_k(x) , \tag{3.16}$$

where $\phi_k(x)$ and their derivatives $\phi_k(x;y_1,\nu_1,\ldots,y_m,\nu_m)$ with respect to the periodic gauge field variables $U(y_1,\nu_1)$, …, $U(y_m,\nu_m)$ satisfy

$$\phi_k(x;y_1,\nu_1,\ldots,y_m,\nu_m) = 0 \qquad \textit{if} \quad 2k < \max_{z=y,y_1,\ldots,y_n} \|x-z\| , \tag{3.17}$$

$$|\phi_k(x;y_1,\nu_1,\ldots,y_m,\nu_m)| \leq d_m k^{q_m} \mathrm{e}^{-k/\varrho} \qquad (m \geq 1) \tag{3.18}$$

for a constant d_m and the integer $q_m \geq 0$. where the constants $d_m, q_m \geq 0$ and the localization range $\varrho > 0$ are independent of the gauge field and of the lattice size L.

Then it follows that

$$\|\phi(x;y_1,\nu_1,\ldots,y_m,\nu_m)\| \leq d'_m (1+\|x-z\|^{q_m}) \mathrm{e}^{-\|x-z\|/\varrho} \qquad (m \geq 1) \tag{3.19}$$

for a constant d'_m, where z is one of $y_1, \ldots, y_m \in \Gamma_{4\,[x]}$ for which $\|x-z\|$ is the maximum.

Thus we adopt the notion of local composite fields on the finite periodic lattice defined by the properties (I) and (II). In the following sections, we will develop a method of the cohomological analysis which directly applies to this kind of local fields on the finite lattice.

[8] For the overlap Dirac operator in the finite-volume lattice, the expansion eq. (3.10) may be the Legendre polynomial expansion with respect to the operator z which involves the square of the Wilson-Dirac operator in the finite-volume lattice,

$$D_{wL}(x,y) = D_w(x,y) + \sum_{n \in \mathbb{Z}^4, n \neq 0} D_w(x, y+nL) . \tag{3.15}$$

With this expansion in the finite-volume lattice, the same argument as in [6] holds true and the locality bound eq. (3.14) follows for the lattice points x and $y, y_1, \ldots, y_n \in \Gamma_{4\,[x]}$.

4. The Poincaré lemma on the finite lattice

In this section, we reformulate the Poincaré lemma given in [29] for the finite volume lattice. We consider the finite lattice of integer vectors $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ in n dimensions where $n \geq 1$ is left unspecified,

$$\Gamma_n = \{x \in \mathbb{Z}^n | -L/2 \leq x_\mu < L/2\}. \tag{4.1}$$

The size of the lattice, L, is assumed to be an even integer for simplicity.

In the following we will be concerned with tensor fields $f_{\mu_1 \cdots \mu_k}(x)$ on Γ_n that are totally anti-symmetric in the indices $\mu_1, \ldots, \mu_k$ which may be regarded as periodic tensor fields on the infinite lattice:

$$f_{\mu_1 \cdots \mu_k}(x + L\hat{\nu}) = f_{\mu_1 \cdots \mu_k}(x) \qquad \text{for all } \mu, \nu = 1, \ldots, n. \tag{4.2}$$

The differential forms on the finite lattice are introduced following [29]. The general k-form on Γ_n is given by

$$f(x) = \frac{1}{k!} f_{\mu_1 \cdots \mu_k}(x) dx_{\mu_1} \cdots dx_{\mu_k}. \tag{4.3}$$

The linear space of all these forms is denoted by Ω_k. An exterior difference operator $\mathrm{d} : \Omega_k \to \Omega_{k+1}$ is defined through

$$df(x) = \frac{1}{k!} \partial_\mu f_{\mu_1, \ldots, \mu_k}(x) dx_\mu dx_{\mu_1} \cdots dx_{\mu_k}, \tag{4.4}$$

where ∂_μ denotes the forward nearest-neighbor difference operator. The associated divergence operator $d^* : \Omega_k \to \Omega_{k-1}$ is defined in the obvious way by setting $d^* f = 0$ if f is a 0-form and

$$d^* f(x) = \frac{1}{(k-1)!} \partial^*_\mu f_{\mu\mu_2 \cdots \mu_k}(x) dx_{\mu_2} \cdots dx_{\mu_k} \tag{4.5}$$

in all other cases, where ∂^*_μ is the backward nearest-neighbor difference operator. With respect to the natural scalar product for tensor fields on Γ_n, d^* is equal to minus the adjoint operator of d.

It is straightforward to show that $d^{*2} = 0$ and the difference equation $d^* f = 0$ is hence solved by all forms $f = d^* g$. In the infinite lattice, the Poincaré lemma [29] asserts that these are in fact all solutions, an exception being the 0-forms where one has a one-dimensional space of further solutions and one needs the extra condition, $\sum_x f(x) = 0$, to remove them. On the finite lattice, the lemma can be formulated so that it holds true up to a certain correction form $\Delta f(x)$ which coefficients are just some linear combinations of the coefficients of $f(x)$ at the boundary of Γ_n.[9]

Lemma 1. *(Modified Poincaré lemma) Let f be a k-form which satisfies*

$$d^* f = 0 \quad \text{and} \quad \sum_{x \in \Gamma_n} f(x) = 0 \quad \text{if} \quad k = 0. \tag{4.6}$$

[9] An equivalent formulation of the Poincaré lemma can be given in terms of divergence operator d.

Then there exist two forms $g \in \Omega_{k+1}$ and $\Delta f \in \Omega_k$ such that

$$f = d^* g + \Delta f \tag{4.7}$$

for certain constants C_3 and $p_3 \geq 0$. The tensor field $\Delta f_{\mu_1 \cdots \mu_k}(x)$ linearly depends only on the values of the tensor field $f_{\mu_1 \cdots \mu_k}(x)$ at the boundary, $\{f_{\mu_1 \cdots \mu_k}(z) |\ z \in \partial\Gamma_n\}$.

On the other hand, in the continuum, it follows from the de Rham theorem that a closed form whose integral over any cycle vanishes identically can be expressed as an exact form. On the finite lattice Γ_n, any exact k-form($k \geq 1$), $f = d^* g$, satisfies

$$\sum_{x_\nu; \nu \neq \mu_1, \ldots, \mu_k} f_{\mu_1 \cdots \mu_k}(x) = 0 \qquad \text{for all } \mu_1, \ldots, \mu_k, \tag{4.8}$$

or, using $\partial^*_\mu f_{\mu\mu_2 \cdots \mu_k}(x) = 0$,

$$\sum_{x \in \Gamma_n} f_{\mu_1 \cdots \mu_k}(x) = 0 \qquad \text{for all } \mu_1, \ldots, \mu_k\,. \tag{4.9}$$

Then we can show the following lemma.

Lemma 2. *(Corollary of de Rham theorem) Let f be a k-form which satisfies*

$$d^* f = 0 \quad \text{and} \quad \sum_{x \in \Gamma_n} f(x) = 0\,. \tag{4.10}$$

Then there exist a form $g \in \Omega_{k+1}$ such that

$$f = d^* g\,. \tag{4.11}$$

The proof of the lemma 1 (lemma 2) goes just like that in [29] with some modifications. We first show that the lemma holds for $n = 1$ and then proceed to higher dimensions by induction. On a one-dimensional lattice the non-trivial forms are the 0- and 1-forms. In the first case we have $k = 0$ and the only condition on f is then that $\sum_{x \in \Gamma_1} f(x) = 0$. It follows from this that the 1-form

$$g(x) = \sum_{y_1 = x_{01} - L/2}^{x_1} f(y)\, dx_1 \tag{4.12}$$

is periodic and satisfies $d^* g = f$. In the other case the equation $d^* f = 0$, that is, $\partial^*_\mu f_\mu(x) = 0$ implies that $f_1(x) =$ constant. Since one may choose x as the point at the boundary $\|x - x_0\| = L/2$, we have $f_1(x) = f_1(z)|_{z \in \partial\Gamma_1}$ which is what the lemma claims. Thus we have proved the lemma for $n = 1$.

(If the one-form f satisfies $\sum_{x_1 \in \Gamma_1} f_1(x_1) = 0$, the constant must vanishes identically and the second version of the lemma holds for $n = 1$.)

Let us now assume that n is greater than 1 and that the lemma holds in dimension $n - 1$. We then decompose the form f according to

$$f = u dx_n + v\,, \tag{4.13}$$

where u and v are elements of Ω_{k-1} and Ω_k respectively that are independent of dx_n.

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If we ignore the dependence on x_n, these forms may be regarded as forms in $n-1$ dimensions. To avoid any confusion the corresponding exterior divergence operator will be denoted by $\bar{d}^*$. It is then straightforward to show that

$$d^*f = (\bar{d}^*u)\,dx_n + \{(-1)^{k-1}\partial_n^* u + \bar{d}^* v\} \tag{4.14}$$

and the equation $d^*f = 0$ hence implies

$$(-1)^{k-1}\partial_n^* u + \bar{d}^* v = 0\,, \qquad \bar{d}^* u = 0\,. \tag{4.15}$$

Note that $u = 0$ if $k = 0$ and the condition eq. (4.6) reduces to $\sum_{x\in\Gamma_n} v(x) = 0$.

We now define a form $\bar{v}$ on Γ_{n-1} through

$$\bar{v}(x) = \sum_{y_n=-L/2}^{L/2-1} v(y)\,, \qquad y = (x_1, \ldots, x_{n-1}, y_n)\,. \tag{4.16}$$

Evidently $\bar{v}$ is an element of Ω_k and from the above one infers that it satisfies the premises of the lemma. The induction hypothesis thus allows us to conclude that $\bar{v} = \bar{d}^*\bar{g} + \Delta\bar{v}$ for some form $\bar{g} \in \Omega_{k+1}$ and some correction form $\Delta\bar{v} \in \Omega_k$ in $n-1$ dimensions. $\Delta\bar{v}$ depends only on $\{\bar{v}(z)|\ z \in \partial\Gamma_{n-1}\}$.

Next we introduce a new form h on Γ_n through

$$h(x) = (-1)^k \sum_{y_n=x_{0n}-L/2}^{x_n} \{v(y) - \delta_{y_n,x_{0n}}\bar{v}\}\,dx_n \tag{4.17}$$

where y is as in eq. (4.16). h is periodic. Using eq. (4.15) it is straightforward to prove that

$$f(x) = \delta_{x_n,x_{0n}}\bar{v}(x) + d^*h(x) + u(x)|_{x_n=x_{0n}-L/2-1}\,dx_n\,. \tag{4.18}$$

$u(x)|_{x_n=x_{0n}+L/2-1}$ may be regarded as the $(k-1)$-form on Γ_{n-1}. To make it explicit, we denote the form as

$$u(x)|_{x_n=x_{0n}+L/2-1} = \bar{u}(x)\,. \tag{4.19}$$

Since the second condition in eq. (4.15) implies that $\bar{d}^*\bar{u} = 0$, it follows

$$\bar{u}(x) = \bar{d}^*\bar{e}(x) + \Delta\bar{u}(x)\,, \tag{4.20}$$

where $\Delta\bar{u}$ depends only on $\{\bar{u}(z)|\ z \in \partial\Gamma_{n-1}\}$. We thus conclude that the sum

$$g(x) = \delta_{x_n,x_{0n}}\bar{g}(x) + h(x) + \bar{e}(x)\,dx_n \tag{4.21}$$

is an element of Ω_{k+1} such that

$$f(x) = d^*g(x) + \Delta f(x) \tag{4.22}$$

where

$$\Delta f(x) = \delta_{x_n,x_{0n}}\Delta\bar{v}(x) + \Delta\bar{u}(x)\,dx_n\,. \tag{4.23}$$

Δf depends only on $\{\bar{v}(z)|\ z \in \Gamma_{n-1}\}$ and $\{\bar{u}(z)|\ z \in \Gamma_{n-1}\}$ and therefore depends only on the boundary values of $f(x)$, $\{f(z)|\ z \in \partial\Gamma_n\}$.

(For the form f satisfying $\sum_{x\in\Gamma_n} f(x)=0$, it follows from eq. (4.18) that $\sum_{x\in\Gamma_{n-1}}\bar{v}(x)=0$ and $\sum_{x\in\Gamma_{n-1}} u(x)|_{x_n=x_{0n}+L/2-1} = \sum_{x\in\Gamma_{n-1}} \bar{u}(x) = 0$. From the induction hypothesis, it follows immediately that $\Delta\bar{v}(x) = 0$ and $\Delta\bar{u}(x) = 0$. Therefore we have $f(x) = d^* g(x)$ and the second version of the lemma holds in dimension n.)

The construction of the forms $g(x)$ and $\Delta f(x)$ is given explicitly in the above proof of the lemma. The coefficients of $g(x)$ and $\Delta f(x)$ are some linear combinations of the coefficients of $f(x)$ and therefore the sizes of these forms are intimately related to that of $f(x)$. Now let us introduce norms of the forms by

$$\|f\|_{x_0,p,\varrho} = \max_{\mu_1\cdots\mu_k, x\in\Gamma_n} \frac{|f_{\mu_1\cdots\mu_k}(x+x_0)|}{(1+\|x\|^p)\,\mathrm{e}^{-\|x\|/\varrho}}\,, \tag{4.24}$$

with a localization range ϱ, an integer p and a reference point x_0 fixed. Then we can show the following bound for the norm of the form $g(x)$:

$$\|g\|_{x_0,p,\varrho} \leq C\,\|f\|_{x_0,p,\varrho} \tag{4.25}$$

for some constant C independent of $f(x)$, x_0 and L. The proof of this bound is given in the appendix A. As for the form $\Delta f(x)$, we have at least

$$|\Delta f_{\mu_1\cdots\mu_k}(x)| \leq n\,C'\,L^{n-1} \max_{\mu_1\cdots\mu_k, z\in\partial\Gamma_n} |f_{\mu_1\cdots\mu_k}(z+x_0)| \tag{4.26}$$

for some constant C' independent of $f(x)$ and L.

In the cohomological analysis of chiral anomaly, we encounter tensor fields which are norm-bounded with a reference point $x_0 \in \Gamma_n$, an integer p and a localization range ϱ as

$$\|f\|_{x_0,p,\varrho} \leq C_1 \tag{4.27}$$

for a constant C_1 independent of L (and also the gauge field). This locality property holds true for the tensor fields which are obtained from the chiral anomaly $q(x)$ defined with the overlap Dirac operator [2, 4, 6] by the differentiation with respect to the link variables. For such tensor fields, the lemmas 1 and 2 imply that the form $g(x)$ should satisfy the bound

$$\|g\|_{x_0,p,\varrho} \leq C_2\,, \tag{4.28}$$

and the form $\Delta f(x)$ should satisfy the more stringent bound than eq. (4.26)

$$|\Delta f_{\mu_1\cdots\mu_k}(x)| \leq C_3\,L^p\,\mathrm{e}^{-L/2\varrho} \tag{4.29}$$

for certain constants C_2, C_3 which do not depend on L (and also the gauge field). Therefore $g(x)$ has the same locality property as that of $f(x)$, and $\Delta f(x)$ is a small finite-volume correction of order $\mathrm{O}(\mathrm{e}^{-L/2\varrho})$.

5. Vector potentials on the finite lattice

In this section, we examine the parametrization of the gauge fields on the finite lattice in terms of vector potentials. We restrict ourselves to the case of four dimensions. It is straightforward to extend the following discussion to other dimensions.

In the course of the argument of the local cohomology problem, it plays a crucial role to introduce the vector potential which has the one-to-one correspondence to the original link variable. In this respect, an important point is that the locality properties of gauge invariant fields should be the same independently of whether they are considered to be functions of the link variable or the vector potential. Similar to the lemma 5.1 of [29], the following lemma shows that for $\tilde{U}(x,\mu)$, the actual local and dynamical degrees of freedom in a given topological sector, it is possible to establish the one-to-one correspondence to a *periodic* vector potential with the desired locality properties on the finite lattice.

Lemma 3. *There exists a periodic vector potential $\tilde{A}_\mu(x)$ such that*

$$\tilde{U}(x,\mu) = \mathrm{e}^{i\tilde{A}_\mu(x)}\,, \tag{5.1}$$

$$F_{\mu\nu}(x) = \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) + \frac{2\pi m_{\mu\nu}}{L^2}\,, \tag{5.2}$$

$$\begin{cases} |\tilde{A}_\mu(x)| \leq \pi(1+4\|x\|) & \text{for} \quad x_\nu \neq \frac{L}{2}-1 \ (\nu=1,2,3), \\ |\tilde{A}_\mu(x)| \leq \pi(1+6L^2) & \text{otherwise.} \end{cases} \tag{5.3}$$

Moreover, if $\tilde{A}'_\mu(x)$ is any other field with these properties we have

$$\tilde{A}'_\mu(x) = \tilde{A}_\mu(x) + \partial_\mu \omega(x)\,, \tag{5.4}$$

where the gauge function $\omega(x)$ takes values that are integer multiples of 2π.

Proof. We introduce a vector potential

$$\tilde{a}_\mu(x) = \frac{1}{i}\ln \tilde{U}(x,\mu)\,, \qquad -\pi < \tilde{a}_\mu(x) \leq \pi \tag{5.5}$$

and then note that

$$F_{\mu\nu}(x) = \partial_\mu \tilde{a}_\nu(x) - \partial_\nu \tilde{a}_\mu(x) + \frac{2\pi m_{\mu\nu}}{L^2} + 2\pi \tilde{n}_{\mu\nu}(x)\,, \tag{5.6}$$

where $\tilde{n}_{\mu\nu}(x)$ is an anti-symmetric tensor field with integer values which satisfies

$$\partial_{[\rho}\tilde{n}_{\mu\nu]}(x) = 0\,, \tag{5.7}$$

$$\sum_{s,t=0}^{L-1} \tilde{n}_{\mu\nu}(x+s\hat{\mu}+t\hat{\nu}) = 0\,. \tag{5.8}$$

The Bianchi identity of $\tilde{n}_{\mu\nu}(x)$ follows from the Bianchi identity of $F_{\mu\nu}(x)$ which holds true for $\epsilon < \pi/3$.

We now construct a *periodic* integer vector field $\tilde{m}_\mu(x)$ such that $\partial_\mu \tilde{m}_\nu - \partial_\nu \tilde{m}_\mu = \tilde{n}_{\mu\nu}$. For this purpose, we try to impose a complete axial gauge where $\tilde{m}_1(x)=0$, $\tilde{m}_2(x)|_{x_1=0} = 0$, $\tilde{m}_3(x)|_{x_1=x_2=0}=0$, $\tilde{m}_4(x)|_{x_1=x_2=x_3=0}=0$ and to obtain the non-zero components of the field by solving

$$\partial_\mu \tilde{m}_\nu(x) = \tilde{n}_{\mu\nu}(x) \qquad \text{at } x_1 = \cdots = x_{\mu-1} = 0 \tag{5.9}$$

for $\mu = 3, 2, 1$ (in this order) and $\nu > \mu$. However, the resulted vector potential is not periodic. Let us denote the restriction of the solution on to Γ_4 by $m_\mu(x)$,

$$m_\mu(x) = -\sum_{\nu<\mu} \sum_{t_\nu=0}^{x_\nu-1}{}' \; \tilde{n}_{\mu\nu}(z^{(\nu)})\Big|_{x_1=\cdots=x_{\nu-1}=0} \tag{5.10}$$

where $x \in \Gamma_4$, $z^{(\nu)} = (x_1, \ldots, t_\nu, \ldots)$ and

$$\sum_{t_i=0}^{x_i-1}{}' f(x) = \begin{cases} \sum_{t_i=0}^{x_i-1} f(x) & (x_i \geq 1) \\ 0 & (x_i = 0) \\ \sum_{t_i=x_i}^{-1} (-1) f(x) & (x_i \leq -1) \end{cases} . \tag{5.11}$$

Although it satisfies the bound $|m_\mu(x)| \leq 2\|x\|$, it only satisfies

$$\tilde{n}_{\mu\nu} = \partial_\mu m_\nu - \partial_\nu m_\mu + \Delta\tilde{n}_{\mu\nu} , \tag{5.12}$$

$$\Delta\tilde{n}_{\mu\nu}(x) = \delta_{x_\mu, L/2-1} \sum_{\tilde{t}_\mu=0}^{L-1} \tilde{n}_{\mu\nu}(z^{(\mu,\nu)})\Big|_{x_{\nu+1}=\cdots=0} , \tag{5.13}$$

where $\nu > \mu$ and $\tilde{t}_\mu = t_\mu \bmod L$. We note that $\Delta\tilde{n}_{\mu\nu}(x)$ has the support on the boundary of Γ_4. We then use the lattice counterpart of the lemma 9.2 in [30], to obtain the periodic integer vector potential $\Delta m_\mu(x)$ which solve $\partial_\mu \Delta m_\nu - \partial_\nu \Delta m_\mu = \Delta\tilde{n}_{\mu\nu}$,

$$\Delta m_\mu(x) = -\delta_{x_\mu, L/2-1} \sum_{\nu>\mu} \sum_{\tilde{t}_\mu=0}^{L-1} \sum_{t_\nu=0}^{x_\nu-1}{}' \; \tilde{n}_{\mu\nu}(z^{(\mu,\nu)})\Big|_{x_{\nu+1}=\cdots=0} . \tag{5.14}$$

The desired periodic integer vector potential $\tilde{m}_\mu(x)$ is now obtained by $\tilde{m}_\mu(x) = m_\mu(x) + \Delta m_\mu(x)$, which satisfies the bound

$$\begin{cases} |\tilde{m}_\mu(x)| \leq 2\|x\| & \text{for } \; x_\nu \neq L/2-1 \; (\nu = 1, 2, 3), \\ |\tilde{m}_\mu(x)| \leq 3L^2 & \text{otherwise.} \end{cases} \tag{5.15}$$

and the vector potential which represents the link variable $\tilde{U}(x, \mu)$ is obtained by

$$\tilde{A}_\mu(x) = \tilde{a}_\mu(x) + 2\pi\tilde{m}_\mu(x) . \tag{5.16}$$

Thus lemma establishes a one-to-one correspondence between the admissible fields $\tilde{U}(x, \mu) \times V_{[m]}(x, \mu)$ and the vector fields $\tilde{A}_\mu(x)$ with field tensor $\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = F_{\mu\nu}(x) - 2\pi m_{\mu\nu}/L^2$ where $F_{\mu\nu}(x)$ is bounded by ϵ.

Locality property of this mapping can be argued as in the case of the lemma 5.1 in [29]. A gauge invariant field composed from the link variables $\tilde{U}(x, \mu)\, V_{[m]}(x, \mu)$ may be regarded as a gauge invariant field depending the vector potential $\tilde{A}_\mu(x)$ and also on the magnetic fluxes $m_{\mu\nu}$. The locality properties of the gauge invariant fields are the same independently of whether they are considered to be functions of the link variables or the vector potential. Since the mapping

$$\tilde{A}_\mu(x) \rightarrow \tilde{U}(x, \mu) = \mathrm{e}^{i\tilde{A}_\mu(x)} \tag{5.17}$$

is manifestly local, this is immediately clear if one starts with a field composed from the link variables. In the other direction, starting from a gauge invariant local field $\phi(y)$ depending on the vector potential (and also on the magnetic fluxes $m_{\mu\nu}$), the key observation is that one is free to change the gauge of the integer field $\tilde{m}_\mu(x)$ in eq. (5.16). In particular, we may impose a complete axial gauge taking the point y as the origin. Around y the vector potential is then locally constructed from the given link field and $\phi(y)$ thus maps to a local function of the link variables residing there.[10]

6. Cohomological analysis of topological fields on the finite lattice

In this section, equipped with the Poincaré lemma reformulated for the finite lattice and the periodic and bounded vector potential representation of the link variables, we now perform a cohomological analysis of topological fields directly on the finite lattice. We consider a gauge-invariant local field $q(x)$ on the finite volume lattice which is topological in the sense that

$$\sum_{x\in\Gamma_4} q(x) = \text{integer} \tag{6.1}$$

and also that

$$\sum_{x\in\Gamma_4} \delta q(x) = 0 \tag{6.2}$$

for any local variation of the gauge field. $q(x)$ can be separated into two parts, the part defined in the infinite lattice $q_\infty(x)$ and the part of finite-volume correction $\Delta_L q(x)$ as

$$q(x) = q_\infty(x) + \Delta q_\infty(x)\,. \tag{6.3}$$

$q_\infty(x)$ is a gauge-invariant local field defined in the infinite lattice, but restricted with periodic gauge fields. We assume that this part is also topological in the sense that

$$\sum_{x\in\mathbb{Z}^4} \delta q_\infty(x) = 0\,, \tag{6.4}$$

for any local variation of the gauge field (not restricted to periodic gauge fields). On the other hand, $\Delta q_\infty(x)$ satisfies the bound

$$|\Delta q_\infty(x)| \le \kappa L^\sigma \mathrm{e}^{-L/2\varrho}\,. \tag{6.5}$$

for a constant κ and an integer σ. ϱ is the localization range of $q(x)$. This property of $q(x)$ allows us to relate our result obtained directly on the finite lattice to that obtained through the cohomological analysis in the infinite lattice [29, 52]. All the above properties of the topological field $q(x)$ are satisfied for the chiral anomaly given in terms of the overlap Dirac operator.

[10] As shown in [30], $\tilde{U}(x,\mu)$ can be parametrized uniquely by the Wilson lines w_μ, the transverse vector potential $A_\mu^T(x)$ and the gauge function $\Lambda(x)$. This parametrization of the link variable, however, does not possess the desired locality properties and does not suit for our purpose.

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6.1 Cohomological analysis of topological field on the finite lattice

Let $q(x)$ be a gauge-invariant local field on the finite lattice which is topological in the sense that

$$\sum_{x\in\Gamma_4} \delta q(x) = 0\,, \tag{6.6}$$

for any local variation of the gauge field. Our aim is then to establish

$$q(x) = q_{[m]}(x) + \phi_{[m]\mu\nu}(x)\tilde{F}_{\mu\nu}(x) + \gamma_{[m,w]}\epsilon_{\mu\nu\rho\sigma}\tilde{F}_{\mu\nu}(x)\,\tilde{F}_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) + \partial_\mu^*\tilde{k}_\mu(x)\,, \tag{6.7}$$

where $q_{[m]}(x)$ denotes the same field for the configuration $V_{[m]}(x,\mu)$, $\phi_{[m]\mu\nu}(x)$ is a gauge invariant functional of $V_{[m]}(x,\mu)$, $\gamma_{[m,w]}$ is a constant which may depend on $V_{[m]}(x,\mu)$ and a constant Wilson line, and $\tilde{k}_\mu(x)$ is a gauge invariant local current. The first step of the proof of eq. (6.7) is the following lemma, which corresponds to the lemma 6.1 in [29].

Lemma 4. *There exist gauge invariant local fields $\phi_{\mu\nu}(x)$ and $h_\mu(x)$ such that*

$$\phi_{\mu\nu}(x) = -\phi_{\nu\mu}(x), \qquad \partial_\mu^*\phi_{\mu\nu}(x) = 0, \tag{6.8}$$

$$q(x) = q_{[m]}(x) + \phi_{\mu\nu}(x)\,\tilde{F}_{\mu\nu}(x) + \partial_\mu^* h_\mu(x)\,. \tag{6.9}$$

Proof. The vector potential $\tilde{A}_\mu(x)$ represents an admissible field through $\mathrm{e}^{i\tilde{A}_\mu(x)}\times V_{[m]}(x,\mu)$ and the associated field tensor $F_{\mu\nu}(x) = \partial_\mu\tilde{A}_\nu(x) - \partial_\nu\tilde{A}_\mu(x) + \frac{2\pi m_{\mu\nu}}{L^2}$ is hence bounded by ϵ. It is straightforward to check that this property is preserved if the potential is scaled by a factor t in the range $0\le t\le 1$, i.e. we can contract the vector potential to zero without leaving the space of admissible fields. Differentiation and integration with respect to t then yields

$$q(x) = q_{[m]}(x) + \sum_{y\in\Gamma_4} j_\nu(x,y)\tilde{A}_\nu(y)\,, \tag{6.10}$$

where

$$j_\nu(x,y) = \int_0^1 dt\left(\frac{\partial q(x)}{\partial\tilde{A}_\nu(y)}\right)_{\tilde{A}\to t\tilde{A}}. \tag{6.11}$$

As a function of the gauge fields, the current $j_\nu(x,y)$ has the same locality properties as the topological field.

Since the gauge group is abelian, the derivative of a gauge invariant field with respect to the vector potential is gauge invariant and the same is hence true for $j_\nu(x,y)$. Performing an infinitesimal gauge transformation in eq. (6.10), it then follows that

$$j_\nu(x,y)\overleftarrow{\partial_\nu^*} = 0\,. \tag{6.12}$$

Here and below the convention is adopted that a difference operator refers to x or y depending on whether it appears on the left or the right of an expression.

The lemma 1 now allows us to conclude that there exists a gauge invariant anti-symmetric tensor field $\theta_{\mu\nu}(x,y)$ such that

$$j_\nu(x,y) = \theta_{\nu\mu}(x,y)\overleftarrow{\partial_\mu^*} + \Delta j_\nu(x,y)\,, \qquad |\Delta j_\nu(x,y)| \le \kappa_1 L^{\sigma_1}\mathrm{e}^{-L/2\rho}\,. \tag{6.13}$$

As explained in section 4, the construction of this field involves a reference point x_0 which is here taken to be x. This choice ensures that $\theta_{\mu\nu}(x,y)$ has the same locality properties as $j_\nu(x,y)$. $\Delta j_\nu(x,y)$ is a small field which satisfies the exponential bound. In the following, the symbol Δ should be understood to denote such exponentially small fields satisfying certain similar bounds.

When eq. (6.13) is inserted in eq. (6.10), a partial summation yields

$$q(x) = q_{[m]}(x) + \frac{1}{2}\sum_{y\in\Gamma_4} \theta_{\mu\nu}(x,y)\,\tilde{F}_{\mu\nu}(y) + \sum_{y\in\Gamma_4} \Delta j_\nu(x,y)\tilde{A}_\nu(y)\,. \tag{6.14}$$

This may be rewritten in the form

$$q(x) = q_{[m]}(x) + \tilde{\phi}_{\mu\nu}(x)\,\tilde{F}_{\mu\nu}(x) + \frac{1}{2}\sum_{y\in\Gamma_4} \eta_{\mu\nu}(x,y)\,\tilde{F}_{\mu\nu}(y) + \sum_{y\in\Gamma_4} \Delta j_\nu(x,y)\tilde{A}_\nu(y)\,, \tag{6.15}$$

where the new fields are given by

$$\tilde{\phi}_{\mu\nu}(x) = \frac{1}{2}\sum_{z\in\Gamma_4} \theta_{\mu\nu}(z,x) \tag{6.16}$$

$$\eta_{\mu\nu}(x,y) = \theta_{\mu\nu}(x,y) - \delta_{x,y}\sum_{z\in\Gamma_4} \theta_{\mu\nu}(z,y)\,. \tag{6.17}$$

Both of them are gauge invariant and anti-symmetric in the indices μ,ν. Moreover, taking the locality properties of $\theta_{\mu\nu}(x,y)$ into account, it is easy to prove that $\tilde{\phi}_{\mu\nu}(x)$ is a local field.

Using eq. (6.13) and the topological properties of $q(x)$ one obtains

$$\partial^*_\mu \tilde{\phi}_{\mu\nu}(x) = \frac{1}{2}\sum_{z\in\Gamma_4} \Delta j_\nu(z,x)\,. \tag{6.18}$$

Since the r.h.s. of eq. (6.18) is an exact form, it satisfies the premise of the lemma 2. Therefore we may apply the lemma 2 to obtain

$$\frac{1}{2}\sum_{z\in\Gamma_4} \Delta j_\nu(z,x) = \partial^*_\mu \Delta\Phi_{\mu\nu}(x)\,. \tag{6.19}$$

We may then define another tensor field

$$\phi_{\mu\nu}(x) \equiv \tilde{\phi}_{\mu\nu}(x) - \Delta\Phi_{\mu\nu}(x)\,, \tag{6.20}$$

which satisfies

$$\partial^*_\mu \phi_{\mu\nu}(x) = 0\,. \tag{6.21}$$

From the redefinition of $\tilde{\phi}_{\mu\nu}(x)$, eq. (6.15) may be rewritten in the form

$$q(x) = q_{[m]}(x) + \phi_{\mu\nu}(x)\,\tilde{F}_{\mu\nu}(x) + \frac{1}{2}\sum_{y\in\Gamma_4} \eta_{\mu\nu}(x,y)\,\tilde{F}_{\mu\nu}(y) + \Delta q(x)\,, \tag{6.22}$$

where

$$\Delta q(x) = \Delta\Phi_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x) + \sum_{y\in\Gamma_4} \Delta j_\nu(x,y)\tilde{A}_\nu(y)\,. \tag{6.23}$$

As for the fields $\eta_{\mu\nu}(x,y), \Delta q(x)$ we note that

$$\sum_{x\in\Gamma_4} \eta_{\mu\nu}(x,y) = 0\,, \qquad \sum_{x\in\Gamma_4} \Delta q(x) = 0 \tag{6.24}$$

and the lemma 1 (or 2) may hence be applied again. This leads to the representation

$$\eta_{\mu\nu}(x,y) = \partial^*_\lambda \tau_{\lambda\mu\nu}(x,y)\,, \qquad \Delta q(x) = \partial^*_\lambda \Delta h_\lambda(x) \tag{6.25}$$

in terms of new fields $\tau_{\lambda\mu\nu}(x,y), \Delta h_\lambda(x)$. Then we define

$$h_\mu(x) = \frac{1}{2}\sum_{y\in\Gamma_4} \tau_{\mu\nu\rho}(x,y)\tilde{F}_{\nu\rho}(y) + \Delta h_\mu(x)\,. \tag{6.26}$$

With these results, eq. (6.22) may be finally rewritten in the form

$$q(x) = q_{[m]}(x) + \phi_{\mu\nu}(x)\,\tilde{F}_{\mu\nu}(x) + \partial^*_\mu h_\mu(x) \tag{6.27}$$

and the lemma has thus been proved.

In the second step of the proof of eq. (6.7) we determine the general form of the field $\phi_{\mu\nu}(x)$ using no other properties than those stated in lemma 4. This step corresponds to the lemma 6.2 in [29].

Lemma 5. *There exists a gauge invariant, local and totally anti-symmetric tensor field $t_{\lambda\mu\nu}(x)$ such that*

$$\phi_{\mu\nu}(x) = \phi_{[m]\mu\nu}(x) + \gamma_{[m,w]}\epsilon_{\mu\nu\rho\sigma}\tilde{F}_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) + \partial^*_\lambda t_{\lambda\mu\nu}(x) + \Delta\phi_{\mu\nu}(x), \tag{6.28}$$

where $\phi_{[m]\mu\nu}(x)$ is the value of $\phi_{\mu\nu}(x)$ at $V_{[m]}(x,\mu)$, and $\gamma_{[m,w]}$ is a constant which may depends on $V_{[m]}(x,\mu)$ and a constant Wilson line. $\Delta\phi_{\mu\nu}(x)$ satisfies the bound $|\Delta\phi_{\mu\nu}(x)| \leq \kappa_2 L^{\sigma_2}\mathrm{e}^{-L/2\rho}$.

Proof. Proceeding as in the proof of lemma 4, it is straightforward to derive a representation analogous to eq. (6.14) for the field $\phi_{\mu\nu}(x)$. Only the locality and gauge invariance of the field are required for this and one ends up with the expression

$$\phi_{\mu\nu}(x) = \phi_{[m]\mu\nu}(x) + \frac{1}{2}\sum_{y\in\Gamma_4} \tilde{\xi}_{\mu\nu\rho\sigma}(x,y)\tilde{F}_{\rho\sigma}(y) + \sum_{y\in\Gamma_4} \Delta j_{\mu\nu\rho}(x,y)\,\tilde{A}_\rho(y)\,, \tag{6.29}$$

where $\phi_{[m]\mu\nu}(x)$ is the value of $\phi_{\mu\nu}(x)$ at $V_{[m]}(x,\mu)$ and the new fields $\tilde{\xi}_{\mu\nu\rho\sigma}(x,y)$ and $\Delta j_{\mu\nu\rho}(x,y)$ are defined through

$$j_{\mu\nu\rho}(x,y) = \int_0^1 dt \left(\frac{\partial\phi_{\mu\nu}(x)}{\partial\tilde{A}_\rho(y)}\right)_{\tilde{A}\to t\tilde{A}}\,, \qquad j_{\mu\nu\rho}(x,y)\overleftarrow{\partial}^*_\rho = 0\,, \tag{6.30}$$

$$j_{\mu\nu\rho}(x,y) = \tilde{\xi}_{\mu\nu\rho\sigma}(x,y)\overleftarrow{\partial}^*_\sigma + \Delta j_{\mu\nu\rho}(x,y)\,. \tag{6.31}$$

As a function of the gauge fields, these new fields appearing in eq. (6.29) are gauge invariant and have the same locality properties as the current $j_{\mu\nu\rho}(x,y)$, that is, as the field $\phi_{\mu\nu}(x)$. From eq. (6.31) it follows that $\tilde{\xi}_{\mu\nu\rho\sigma}(x,y)$ satisfies

$$\tilde{\xi}_{\mu\nu\rho\sigma} = -\tilde{\xi}_{\nu\mu\rho\sigma} = -\tilde{\xi}_{\mu\nu\sigma\rho}\,, \tag{6.32}$$

$$\partial^*_\mu \tilde{\xi}_{\mu\nu\rho\sigma}(x,y)\overleftarrow{\partial}^*_\sigma = -\partial^*_\mu \Delta j_{\mu\nu\rho}(x,y)\,. \tag{6.33}$$

Since the r.h.s. of the second equation of eq. (6.33) is an exact form in terms of y, it satisfies the premise of the lemma 2. Therefore we may apply the lemma 2 to obtain

$$-\partial^*_\mu \Delta j_{\mu\nu\rho}(x,y) = \partial^*_\mu \Delta \Xi_{\mu\nu\rho\sigma}(x,y)\overleftarrow{\partial}^*_\sigma. \tag{6.34}$$

Here we note that ∂^*_μ can be extracted explicitly in the r.h.s. of eq. (6.34). This is because $\partial^*_\mu \Delta j_{\mu\nu\lambda}(x,y)$ is already an exponentially small field and when applying the lemma 2 with respect to y, the reference point x_0 is not necessarily identified with x. We may then define another tensor field

$$\xi_{\mu\nu\rho\sigma}(x,y) = \tilde{\xi}_{\mu\nu\rho\sigma}(x,y) - \Delta\Xi_{\mu\nu\rho\sigma}(x,y)\,, \tag{6.35}$$

which satisfies

$$\partial^*_\mu \xi_{\mu\nu\rho\sigma}(x,y)\overleftarrow{\partial}^*_\sigma = 0\,. \tag{6.36}$$

Applying the lemma 1 to eq. (6.36) we have

$$\partial^*_\mu \xi_{\mu\nu\rho\sigma}(x,y) = \tilde{\upsilon}_{\nu\rho\sigma\tau}(x,y)\overleftarrow{\partial}^*_\tau + \Delta\{\partial^*_\mu \xi_{\mu\nu\rho\sigma}\}(x,y)\,, \tag{6.37}$$

$$\partial^*_\nu \tilde{\upsilon}_{\nu\rho\sigma\tau}(x,y)\overleftarrow{\partial}^*_\tau = -\partial^*_\nu \Delta\{\partial^*_\mu \xi_{\mu\nu\rho\sigma}\}(x,y)\,. \tag{6.38}$$

From eq. (6.38), we may again apply the lemma 2 to obtain

$$-\partial^*_\nu \Delta\{\partial^*_\mu \xi_{\mu\nu\rho\sigma}\}(x,y) = \partial^*_\nu \Delta\Upsilon_{\nu\rho\sigma\tau}(x,y)\overleftarrow{\partial}^*_\tau\,. \tag{6.39}$$

We may then define another tensor field

$$\upsilon_{\nu\rho\sigma\tau}(x,y) = \tilde{\upsilon}_{\nu\rho\sigma\tau}(x,y) - \Delta\Upsilon_{\nu\rho\sigma\tau}(x,y), \tag{6.40}$$

which satisfies

$$\partial^*_\nu \upsilon_{\nu\rho\sigma\tau}(x,y)\overleftarrow{\partial}^*_\tau = 0\,. \tag{6.41}$$

Applying the lemma 1 to eq. (6.41) we have

$$\partial^*_\nu \upsilon_{\nu\rho\sigma\tau}(x,y) = \tilde{\omega}_{\rho\sigma\tau\lambda}(x,y)\overleftarrow{\partial}^*_\lambda + \Delta\{\partial^*_\nu \upsilon_{\nu\rho\sigma\tau}\}(x,y)\,, \tag{6.42}$$

$$\sum_{x\in\Gamma_4} \tilde{\omega}_{\rho\sigma\tau\lambda}(x,y)\overleftarrow{\partial}^*_\lambda = -\sum_{x\in\Gamma_4} \Delta\{\partial^*_\nu \upsilon_{\nu\rho\sigma\tau}\}(x,y)\,. \tag{6.43}$$

Since the r.h.s. of eq. (6.43) satisfies the premise of lemma 2. Repeatedly, we may apply the lemma 2 to obtain

$$-\sum_{x\in\Gamma_4} \Delta\{\partial^*_\nu \upsilon_{\nu\rho\sigma\tau}\}(x,y) = \Delta\Omega_{\rho\sigma\tau\lambda}(y)\overleftarrow{\partial}^*_\lambda\,. \tag{6.44}$$

We may then define another tensor field

$$\omega_{\rho\sigma\tau\lambda}(x,y) = \tilde{\omega}_{\rho\sigma\tau\lambda}(x,y) - \delta_{x,y}\,\Delta\Omega_{\rho\sigma\tau\lambda}(y)\,, \tag{6.45}$$

which satisfies

$$\sum_{x\in\Gamma_4} \omega_{\rho\sigma\tau\lambda}(x,y)\overleftarrow{\partial_\lambda^*} = 0\,. \tag{6.46}$$

An immediate consequence of the last equation is that

$$2\gamma_{[m,w]}\,\epsilon_{\rho\sigma\tau\lambda} = \sum_{z\in\Gamma_4} \omega_{\rho\sigma\tau\lambda}(z,x) \tag{6.47}$$

is independent of x. In view of the locality properties of the expression, a dependence on the vector potential is almost excluded except the Wilson line w_μ. It also depends on the magnetic flux $m_{\mu\nu}$ and the size of the lattice L. We will discuss this point later in relation to the anomaly cancellation.

Another application of the lemma 1 (or 2) now implies that

$$\omega_{\rho\sigma\tau\lambda}(x,y) = 2\gamma_{[m,w]}\,\epsilon_{\rho\sigma\tau\lambda}\,\delta_{x,y} + \partial_\nu^*\varphi_{\nu\rho\sigma\tau\lambda}(x,y) \tag{6.48}$$

for some vector field (with respect to x), $\varphi_{\nu\rho\sigma\tau\lambda}(x,y)$. If we define

$$\check{\upsilon}_{\nu\rho\sigma\tau}(x,y) = \upsilon_{\nu\rho\sigma\tau}(x,y) - \varphi_{\nu\rho\sigma\tau\lambda}(x,y)\overleftarrow{\partial_\lambda^*}\,, \tag{6.49}$$

it is then straightforward to prove the relations

$$\partial_\mu^*\xi_{\mu\nu\rho\sigma}(x,y) = \check{\upsilon}_{\nu\rho\sigma\tau}(x,y)\overleftarrow{\partial_\tau^*} + \Delta\{\partial_\mu^*\xi_{\mu\nu\rho\sigma}\}(x,y)\,, \tag{6.50}$$

$$\partial_\nu^*\check{\upsilon}_{\nu\rho\sigma\tau}(x,y) = \left\{2\gamma_{[m,w]}\epsilon_{\rho\sigma\tau\lambda}\,\delta_{x,y}\right\}\overleftarrow{\partial_\lambda^*} + \{\delta_{x,y}\Delta\Omega_{\rho\sigma\tau\lambda}(y)\}\overleftarrow{\partial_\lambda^*} + \Delta\{\partial_\nu^*\upsilon_{\nu\rho\sigma\tau}\}(x,y)\,. \tag{6.51}$$

Compared to eqs. (6.37), (6.42) the important difference is that the form of the first term in the right hand side of the second equation is now known precisely.

In the next step we propagate this information to the first equation by noting

$$\delta_{x,y}\partial_\lambda^* = -\partial_\nu^*\{\delta_{\nu\lambda}\delta_{x,y-\hat{\nu}}\}\,, \tag{6.52}$$

$$\sum_{x\in\Gamma_4}\Delta\{\partial_\nu^*\upsilon_{\nu\rho\sigma\tau}\}(x,y) = 0\,. \tag{6.53}$$

The general solution of eq. (6.51) is hence given by

$$\check{\upsilon}_{\nu\rho\sigma\tau}(x,y) = -\delta_{\nu\lambda}\delta_{x,y-\hat{\nu}}2\gamma_{[m,w]}\epsilon_{\rho\sigma\tau\lambda} + \partial_\mu^*\theta_{\mu\nu\rho\sigma\tau}(x,y) + \Delta\check{\upsilon}_{\nu\rho\sigma\tau}(x,y)\,, \tag{6.54}$$

where $\theta_{\mu\nu\rho\sigma\tau} = -\theta_{\nu\mu\rho\sigma\tau}$. It follows from this that the shifted field

$$\check{\xi}_{\mu\nu\rho\sigma}(x,y) = \xi_{\mu\nu\rho\sigma}(x,y) - \theta_{\mu\nu\rho\sigma\tau}(x,y)\overleftarrow{\partial_\tau^*} \tag{6.55}$$

satisfies the relation

$$\partial_\mu^*\check{\xi}_{\mu\nu\rho\sigma}(x,y) = -2\gamma_{[m,w]}\delta_{\nu\lambda}\delta_{x,y-\hat{\nu}}\overleftarrow{\partial_\tau^*}\epsilon_{\rho\sigma\tau\lambda} + \Delta\{\partial_\mu^*\check{\xi}_{\mu\nu\rho\sigma}\}(x,y)\,, \tag{6.56}$$

where

$$\Delta\{\partial^*_\mu \check{\xi}_{\mu\nu\rho\sigma}\}(x,y) = \Delta\{\partial^*_\mu \xi_{\mu\nu\rho\sigma}\}(x,y) + \Delta\check{v}_{\nu\rho\sigma\tau}(x,y)\overleftarrow{\partial^*_\tau}. \tag{6.57}$$

We may now again use the identity eq. (6.52) and the lemma 1 (or 2) to infer that

$$\check{\xi}_{\mu\nu\rho\sigma}(x,y) = 2\gamma_{[m,w]}\epsilon_{\mu\nu\rho\sigma}\delta_{x,y-\hat{\mu}-\hat{\nu}} + \partial^*_\lambda \kappa_{\lambda\mu\nu\rho\sigma}(x,y) + \Delta\check{\xi}_{\mu\nu\rho\sigma}(x,y) \tag{6.58}$$

where $\kappa_{\lambda\mu\nu\rho\sigma}(x,y)$ and $\Delta\check{\xi}_{\mu\nu\rho\sigma}(x,y)$ are another tensor fields.

Together with

$$\begin{aligned}\phi_{\mu\nu}(x) \;=\; & \phi_{[m]\mu\nu}(x) + \frac{1}{2}\sum_{y\in\Gamma_4} \check{\xi}_{\mu\nu\rho\sigma}(x,y)\tilde{F}_{\rho\sigma}(y) + \frac{1}{2}\sum_{y\in\Gamma_4} \Delta\Xi_{\mu\nu\rho\sigma}(x,y)\tilde{F}_{\rho\sigma}(y) + \\ & + \sum_{y\in\Gamma_4} \Delta j_{\mu\nu\rho}(x,y)\,\tilde{A}_\rho(y)\,, \end{aligned} \tag{6.59}$$

and the definitions

$$t_{\lambda\mu\nu}(x) \;=\; \frac{1}{2}\sum_{y\in\Gamma_4} \kappa_{\lambda\mu\nu\rho\sigma}(x,y)\tilde{F}_{\rho\sigma}(y)\,, \tag{6.60}$$

$$\begin{aligned}\Delta\phi_{\mu\nu}(x) \;=\; & \frac{1}{2}\sum_{y\in\Gamma_4} \Delta\check{\xi}_{\mu\nu\rho\sigma}(x,y)\tilde{F}_{\rho\sigma}(y) + \frac{1}{2}\sum_{y\in\Gamma_4} \Delta\Xi_{\mu\nu\rho\sigma}(x,y)\tilde{F}_{\rho\sigma}(y) + \\ & + \sum_{y\in\Gamma_4} \Delta j_{\mu\nu\rho}(x,y)\,\tilde{A}_\rho(y)\,, \end{aligned} \tag{6.61}$$

this proves the lemma.

The combination of lemma 4 and 5 leads to the representation

$$\begin{aligned} q(x) \;=\; & q_{[m]}(x) + \phi_{[m]\mu\nu}(x)\,\tilde{F}_{\mu\nu}(x) + \gamma_{[m,w]}\epsilon_{\mu\nu\rho\sigma}\tilde{F}_{\mu\nu}(x)\tilde{F}_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) + \partial^*_\mu h_\mu(x) + \\ & + \partial^*_\lambda t_{\lambda\mu\nu}(x)\tilde{F}_{\mu\nu}(x) + \Delta\phi_{\mu\nu}(x)\,\tilde{F}_{\mu\nu}(x)\,. \end{aligned} \tag{6.62}$$

Using the anti-symmetry of the tensor field $t_{\lambda\mu\nu}(x)$ and the vanishing of the monopole current, $\epsilon_{\mu\nu\rho\sigma}\partial_\nu \tilde{F}_{\rho\sigma}(x) = 0$, it is easy to check that

$$\partial^*_\mu t_{\mu\nu\rho}(x)\tilde{F}_{\nu\rho}(x) = \partial^*_\mu \left\{ t_{\mu\nu\rho}(x)\tilde{F}_{\nu\rho}(x+\hat{\mu}) \right\}. \tag{6.63}$$

Moreover it follows that

$$\sum_{x\in\Gamma_4} \Delta\phi_{\mu\nu}(x)\,\tilde{F}_{\mu\nu}(x) = 0\,. \tag{6.64}$$

Since the tensor fields in the r.h.s. are exponentially small, we may apply the lemma 2 with an arbitrary reference point x_0 to obtain

$$\Delta\phi_{\mu\nu}(x)\,\tilde{F}_{\mu\nu}(x) = \partial^*_\mu \Delta k_\mu(x)\,. \tag{6.65}$$

This proves eq. (6.7) with the definition of $\tilde{k}_\mu(x)$ as

$$\tilde{k}_\mu(x) = h_\mu(x) + t_{\mu\nu\rho}(x)\tilde{F}_{\nu\rho}(x+\hat{\mu}) + \Delta k_\mu(x)\,. \tag{6.66}$$

6.2 Relation to the result obtained in the infinite lattice

We will next compare our result eq. (6.7) with the result obtained through the cohomological analysis in the infinite lattice [29, 52]. The later result may be summarized as follows: let us assume that the topological field $q(x)$ on the finite-volume lattice can be separated into two parts, the part defined in the infinite lattice and the part of finite-volume correction,

$$q(x) = q_\infty(x) + \Delta q_\infty(x) . \tag{6.67}$$

$q_\infty(x)$ is the topological field defined in the infinite lattice and with the periodic link variables. $\Delta q_\infty(x)$ satisfies the bound

$$|\Delta q \infty(x)| \leq \kappa L^\sigma \mathrm{e}^{-L/2\varrho} \tag{6.68}$$

for a constant κ and an integer ν. ϱ is the localization range of $q(x)$. Then, through the cohomological analysis in the infinite lattice [29], the first part can be expressed as

$$q_\infty(x) = \alpha + \beta_{\mu\nu} F_{\mu\nu}(x) + \gamma \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x)\, F_{\rho\sigma}(x+\hat\mu+\hat\nu) + \partial^*_\mu k_{\mu\infty}(x) . \tag{6.69}$$

The part of the finite-volume correction is an exponentially small field and it cannot contribute to the topological charge for a sufficiently large L. Namely we have

$$\sum_{x\in\Gamma_4} \Delta q_\infty(x) = 0 . \tag{6.70}$$

Then it can be expressed as the total-divergence of a certain gauge-invariant current which is also exponentially small [52],

$$\Delta q_\infty(x) = \partial^*_\mu \Delta k_{\mu\infty}(x) . \tag{6.71}$$

The total topological field can be expressed as

$$q(x) = \alpha + \beta_{\mu\nu} F_{\mu\nu}(x) + \gamma \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x)\, F_{\rho\sigma}(x+\hat\mu+\hat\nu) + \partial^*_\mu k_\mu(x) , \tag{6.72}$$

where

$$k_\mu(x) \equiv k_{\mu\infty}(x) + \Delta k_{\mu\infty}(x) . \tag{6.73}$$

Since $F_{\mu\nu}(x) = \frac{2\pi m_{\mu\nu}}{L^2} + \tilde F_{\mu\nu}(x)$, the cohomologically non-trivial part of the above result may be rewritten as

$$\begin{aligned} q(x) \,=\, & \alpha + \beta_{\mu\nu}\frac{2\pi m_{\mu\nu}}{L^2} + \gamma\epsilon_{\mu\nu\rho\sigma}\frac{2\pi m_{\mu\nu}}{L^2}\frac{2\pi m_{\rho\sigma}}{L^2} + \\ & + \beta_{\mu\nu}\tilde F_{\mu\nu}(x) + 2\,\gamma\epsilon_{\mu\nu\rho\sigma}\frac{2\pi m_{\rho\sigma}}{L^2}\,\tilde F_{\mu\nu}(x) + \\ & + \gamma\epsilon_{\mu\nu\rho\sigma}\tilde F_{\mu\nu}(x)\,\tilde F_{\rho\sigma}(x+\hat\mu+\hat\nu) + \partial^*_\mu k'_\mu(x) , \end{aligned} \tag{6.74}$$

where

$$k'_\mu(x) = k_\mu(x) + \sum_{\nu\rho\sigma} \gamma\epsilon_{\mu\nu\rho\sigma}\frac{2\pi m_{\mu\nu}}{L^2}\left(F_{\rho\sigma}(x+\hat\mu+\hat\nu) + F_{\rho\sigma}(x+\hat\mu)\right) . \tag{6.75}$$

This result should be compared with eq. (6.7). Indeed, it is possible to show that the coefficients of the polynomials of field tensor $\tilde F_{\mu\nu}(x)$ in eq. (6.7), $q_{[m]}(x)$, $\phi_{[m]\mu\nu}(x)$ and $\gamma_{[m,w]}$, are related to α, $\beta_{\mu\nu}$ and γ by the following bounds.

Lemma 6.

$$\left| q_{[m]}(x) - \alpha - \beta_{\mu\nu} \frac{2\pi m_{\mu\nu}}{L^2} - \gamma \epsilon_{\mu\nu\rho\sigma} \frac{(2\pi)^2 m_{\mu\nu} m_{\rho\sigma}}{L^4} \right| \leq \kappa_3 L^{\sigma_3} \mathrm{e}^{-L/2\rho} , \tag{6.76}$$

$$\left| \phi_{[m]\mu\nu}(x) - \beta_{\mu\nu} - 2\gamma \epsilon_{\mu\nu\rho\sigma} \frac{2\pi m_{\rho\sigma}}{L^2} \right| \leq \kappa_3' L^{\sigma_3} \mathrm{e}^{-L/2\rho} , \tag{6.77}$$

$$\left| \gamma_{[m,w]} - \gamma \right| \leq \kappa_3'' L^{\sigma_3} \mathrm{e}^{-L/2\rho} . \tag{6.78}$$

The proof of the bounds is given in the appendix C.

7. Exact cancellation of gauge anomaly

We now consider the chiral anomaly which is given in terms of the overlap Dirac operator which satisfies the Ginsparg-Wilson relation as follows:

$$q(x) = \mathrm{tr}\left\{ \gamma_5 (1 - D_L)(x,x) \right\} , \tag{7.1}$$

where $D_L(x,y)$ is the finite-volume kernel of the Dirac operator. $q(x)$ is then defined for all admissible gauge fields and it is topological in the sense that

$$\sum_{x \in \Gamma_4} q(x) = \text{integer} \tag{7.2}$$

and also that

$$\sum_{x \in \Gamma_4} \delta q(x) = 0 , \tag{7.3}$$

for any local variation of the gauge field. This chiral anomaly $q(x)$ can be separated into two parts, the part defined in the infinite lattice and the part of finite-volume correction,

$$q(x) = q_\infty(x) + \Delta q_\infty(x) . \tag{7.4}$$

Using eq. (3.1), we explicitly have

$$q_\infty(x) = \mathrm{tr}\left\{ \gamma_5 (1 - D)(x,x) \right\} , \tag{7.5}$$

$$\Delta q_\infty(x) = \sum_{n \in \mathbb{Z}^4, n \neq 0} \mathrm{tr}\left\{ \gamma_5 (1 - D)(x, x + nL) \right\} . \tag{7.6}$$

$q_\infty(x)$ is the chiral anomaly defined in the infinite lattice with the periodic link variables. On the other hand, $\Delta_L q(x)$ satisfies the bound,

$$|\Delta q_\infty(x)| \leq \kappa \, \mathrm{e}^{-L/\varrho} . \tag{7.7}$$

Since $q_\infty(x)$ transforms as pseudo scalar, then we can show that the coefficients α, $\beta_{\mu\nu}$ vanish identically. Moreover γ can be evaluated in perturbation theory, giving the result $\gamma = \frac{1}{32\pi^2}$ [38].

Based on the result obtained in the previous subsections, we will now examine the cancellation of the gauge anomaly. Let us consider an anomaly-free multiplet $\{\psi_L^\alpha(x)\,|\,\alpha = 1,2,\ldots\}$ with the U(1) charges satisfying the condition

$$\sum_\alpha e_\alpha^3 = 0\,. \tag{7.8}$$

In each contribution of a single Weyl fermion $\psi_L^\alpha(x)$ to the gauge anomaly, which is denoted by $q^\alpha(x)$, we scale the vector potential and the magnetic fluxes as

$$\tilde{A}_\mu(x) \to e_\alpha \tilde{A}_\mu(x)\,, \qquad m_{\mu\nu} \to e_\alpha m_{\mu\nu}\,, \tag{7.9}$$

and consider the summation of them in the anomaly-free multiplet,

$$\sum_\alpha e_\alpha\, q^\alpha(x)\,. \tag{7.10}$$

In particular, we consider the summation of the first three terms in the r.h.s. of eq. (6.7),

$$\begin{aligned}\sum_\alpha e_\alpha A^\alpha(x) = \sum_\alpha \Big\{ & e_\alpha q^\alpha_{[m]}(x) + e_\alpha^2 \phi^\alpha_{[m]\mu\nu}(x) \tilde{F}_{\mu\nu}(x) + \\ & + e_\alpha^3 \gamma^\alpha_{[m,w]} \epsilon_{\mu\nu\rho\sigma} \tilde{F}_{\mu\nu}(x)\, \tilde{F}_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) \Big\}\,,\end{aligned} \tag{7.11}$$

Then we can show the following lemma.

Lemma 7. *For an anomaly-free multiplet, the summation of the gauge anomalies $\sum_\alpha e_\alpha A^\alpha(x)$ satisfies the bound*

$$\left| \sum_\alpha e_\alpha A^\alpha(x) \right| \le \kappa_4 L^{\sigma_4} \mathrm{e}^{-L/2\rho}\,, \tag{7.12}$$

and can be written as the total-divergence of a gauge-invariant local current,

$$\sum_\alpha e_\alpha A^\alpha(x) = \partial_\mu^* \Delta \tilde{k}_\mu(x)\,. \tag{7.13}$$

Proof. From the bounds (6.76), (6.77) and (6.78), $e_\alpha A^\alpha(x)$ scales as e_α^3 up to corrections of order $\mathcal{O}(L^\sigma \mathrm{e}^{-L/2\varrho})$. The terms which scale as e_α^3 cancel each other due to the anomaly-free condition and the remaining term satisfies the bound eq. (7.12). Since $\sum_\alpha e_\alpha Q^\alpha = 0$ where $Q^\alpha = \sum_{x\in\Gamma_4} q^\alpha(x) = e_\alpha^2 \gamma \epsilon_{\mu\nu\rho\sigma} (2\pi)^2 m_{\mu\nu} m_{\rho\sigma}$, $\sum_\alpha e_\alpha A^\alpha(x)$ satisfies

$$\sum_{x\in\Gamma_4} \left\{ \sum_\alpha e_\alpha A^\alpha(x) \right\} = 0\,. \tag{7.14}$$

Then we may apply the lemma 2 with an arbitrary reference point x_0 to obtain eq. (7.13). This proves the lemma.

By the combination of lemma 4, 5 and 7, we finally showed that the gauge anomaly is cohomologically trivial. Namely,

$$\sum_\alpha e_\alpha\, q^\alpha(x) = \partial_\mu^* k_\mu(x)\,, \tag{7.15}$$

where $k_\mu(x)$ is a gauge-invariant local current defined by

$$k_\mu(x) = \tilde{k}_\mu(x) + \Delta\tilde{k}_\mu(x) \,. \tag{7.16}$$

Thus we have shown that the current $k_\mu(x)$ which gives the the cohomologically trivial part of the chiral anomaly can be obtained directly from the quantities calculable on the finite lattice. It is in sharp contrast with the current $k_\mu(x)$ given by eq. (6.73) which has been obtained throught the cohomological analysis in the infinite lattice. This is the main result of our analysis.

8. Discussion: Weyl fermion measure on the finite lattice

Finally we will argue how to construct the measure term [30] directly from the cohomologically trivial part of the chiral anomaly on the finite lattice.

In the topological sector with the magnetic flux $m_{\mu\nu}$, we choose an one-parameter family of the admissible U(1) gauge fields as

$$U^t(x,\mu) = \mathrm{e}^{it\tilde{A}_\mu(x)} \times V_{[m]}(x,\mu) \,, \qquad 0 \le t \le 1 \,. \tag{8.1}$$

Let $\eta_\mu(x)$ be a real periodic vector field as a variational parameter. Then we consider the following linear functional on the finite lattice [30, 34]:

$$\begin{aligned} \mathfrak{L}_\eta = i \int_0^1 dt \,\mathrm{Tr}_L \left\{ \hat{P}_L[\partial_t \hat{P}_L, \delta_\eta \hat{P}_L] \right\} + \\ + \int_0^1 dt \sum_{x\in\Gamma_4} \left\{ \eta_\mu(x)\bar{k}_\mu(x) + \tilde{A}_\mu(x)\delta_\eta \bar{k}_\mu(x) \right\} , \end{aligned} \tag{8.2}$$

where $\bar{k}_\mu(x)$ is a gauge-invariant local current, which transforms as an axial vector field under the lattice symmetries and which satisfies $\partial^*_\mu \bar{k}_\mu(x) = \sum_\alpha e_\alpha q^\alpha(x)$. Such a current can be constructed from the current $k_\mu(x)$ in eq. (7.15) by averaging over the lattice symmetries, with the appropriate weights so at to project to the axial current component. This linear functional has the same form as $\mathfrak{L}^\star_\eta$ in [30], the measure term in the infinite volume. We can show that $\mathfrak{L}_\eta$ defined by eq. (8.2) satisfies all the properties required for the measure term *on the finite lattice*.

Lemma 8. *The linear functional $\mathfrak{L}_\eta = \sum_{x\in\Gamma_4} \eta_\mu(x) j_\mu(x)$ defined above has the following properties.*

1. *$j_\mu(x)$ is a local current, which is defined for all admissible gauge fields and depends smoothly on the link variables.*

2. *$j_\mu(x)$ is gauge-invariant and transforms as an axial vector current under the lattice symmetries.*

3. *The linear functional $\mathfrak{L}_\eta$ is a solution of the integrability condition*

$$\delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta = i\mathrm{Tr}_L \left\{ \hat{P}_L[\delta_\eta \hat{P}_L, \delta_\zeta \hat{P}_L] \right\} \tag{8.3}$$

for all periodic variations $\eta_\mu(x)$ and $\zeta_\mu(x)$.

4. *The anomalous conservation law $\partial_\mu^* j_\mu(x) = \sum_\alpha e_\alpha q^\alpha(x)$ holds.*

The proof of the lemma goes just like the proof of the theorem 5.3 in [30] (section 6) and we omit it here. A non-trivial point is the smoothness with respect to the link variables. This is because the periodic vector potentials $\tilde{A}_\mu(x)$ do not cover smoothly the space of the admissible gauge fields and at the singular points, the vector potentials can differ by the gauge functions $\omega(x)$ that are bounded polynomially in the lattice size L. However, $\mathfrak{L}_\eta$ is invariant under gauge transformations $\tilde{A}_\mu(x) \to \tilde{A}_\mu(x) + \partial_\mu \omega(x)$, for arbitrary gauge functions $\omega(x)$ that are bounded polynomially in the lattice size L. Then it can be regarded as a smooth function of the link variables.

The global integrability condition can be also established following the proof of the theorem 5.1 in [30] (section 10). Therefore the linear functional $\mathfrak{L}_\eta = \sum_{x \in \Gamma_4} \eta_\mu(x) j_\mu(x)$ defined above provides a solution to the measure term on the finite lattice.

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A. The bound on $\|g\|$

In this appendix, we give a proof of the bound eq. (4.25). For simplicity, we set $x_0 = 0$. We prove the bound by induction. For $n = 1$, the 0-form g vanishes identically and the bound is satisfied trivially. The 1-form g is given by

$$g(x) = \sum_{y_1=-L/2}^{x_1} f(y)\, dx_1 = \begin{cases} -\sum_{y_1=x_1+1}^{L/2-1} f(y)\, dx_1 & (x_1 \geq 0) \\ \sum_{y_1=-L/2}^{x_1} f(y)\, dx_1 & (x_1 < 0) \end{cases} . \tag{A.1}$$

Then, using $|f(x)| \leq \|f\|_{p,\varrho}(1 + \|x\|^p)\, \mathrm{e}^{-\|x\|/\varrho}$, we can estimate the absolute value of the coefficient of g as

$$\begin{aligned} |g_1(x)| &\leq \begin{cases} \|f\|_{p,\varrho} \sum_{y_1=x_1+1}^{L/2-1} (1 + \|y\|^p)\, \mathrm{e}^{-\|y\|/\varrho} & (x_1 \geq 0) \\ \|f\|_{p,\varrho} \sum_{y_1=-L/2}^{x_1} (1 + \|y\|^p)\, \mathrm{e}^{-\|y\|/\varrho} & (x_1 < 0) \end{cases} \\ &\leq \|f\|_{p,\varrho} \sum_{k=\|x\|}^{\infty} (1 + k^p)\, \mathrm{e}^{-k/\varrho} \\ &\leq \|f\|_{p,\varrho}\, C_{p,\rho}(1 + \|x\|^p)\, \mathrm{e}^{-\|x\|/\varrho}, \end{aligned} \tag{A.2}$$

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where $C_{p,\rho} = \left[\sum_{l=0}^{p} {}_pC_l \frac{\partial^l}{\partial\xi^l}\frac{1}{(1-\mathrm{e}^{\xi})}\right]_{\xi=-1/\varrho}$ which is independent of L. Thus we have the bound on the norm of g as

$$\|g\|_{p,\varrho} \leq C_{p,\varrho}\, \|f\|_{p,\varrho}\,. \tag{A.3}$$

Next we assume the bound eq. (A.3) holds true for $n-1$-dimensions ($n > 1$) and consider the case of n-dimensions. For the closed $k+1$-form $f = u\, dx_n + v$, the k-form g is given by the formula

$$g(x) = \delta_{x_n,0}\, \bar{g}(x) + h(x) + \bar{e}(x)\, dx_n\,, \tag{A.4}$$

where $\bar{g}$ is obtained as

$$\bar{v} = \bar{d}^* \bar{g} + \Delta \bar{v}\,; \qquad \bar{v}(x) \equiv \sum_{y_n=-L/2}^{L/2-1} v(y)\,, \tag{A.5}$$

h is defined by

$$h(x) = (-1)^k \sum_{y_n=-L/2}^{x_n} \{v(y) - \delta_{y_n,0}\, \bar{v}\}\; dx_n = \begin{cases} (-1)^{k+1} \displaystyle\sum_{y_n=x_n+1}^{L/2-1} v(y)\, dx_n & (x_n \geq 0) \\ (-1)^k \displaystyle\sum_{y_n=-L/2}^{x_n} v(y)\, dx_n & (x_n < 0) \end{cases}, \tag{A.6}$$

and $\bar{e}$ is obtained as

$$\bar{u}(x) = \bar{d}^* \bar{e}(x) + \Delta \bar{u}(x)\,; \qquad \bar{u}(x) \equiv u(x)|_{x_n=L/2-1}\,. \tag{A.7}$$

Then the norm of g is bounded by the sum of the norms of $\bar{g}$, h and $\bar{e}$ and therefore we need to evaluate these three norms.

The absolute value of the coefficient of h can be estimated in just the same manner as the case of g for $n = 1$:

$$\begin{aligned} |h_{\mu_1\ldots\mu_{k+1}}(x)| &\leq \begin{cases} \|f\|_{p,\varrho} \displaystyle\sum_{y_n=x_n+1}^{L/2-1} (1+\|y\|^p)\, \mathrm{e}^{-\|y\|/\varrho} & (x_n \geq 0) \\ \|f\|_{p,\varrho} \displaystyle\sum_{y_n=-L/2}^{x_n} (1+\|y\|^p)\, \mathrm{e}^{-\|y\|/\varrho} & (x_n < 0) \end{cases} \\ &\leq \|f\|_{p,\varrho} \sum_{k=\|x\|}^{\infty} (1+k^p)\, \mathrm{e}^{-k/\varrho} \\ &\leq \|f\|_{p,\varrho}\, C_{p,\rho} (1+\|x\|^p)\, \mathrm{e}^{-\|x\|/\varrho}. \end{aligned} \tag{A.8}$$

Thus we have the bound on the norm of h as

$$\|h\|_{p,\varrho} \leq C_{p,\varrho}\, \|f\|_{p,\varrho}\,. \tag{A.9}$$

As to the absolute value of $\bar{v}$, we have an estimate,

$$
\begin{aligned}
|\bar{v}_{\mu_1\ldots\mu_k}(x)| &\leq \|f\|_{p,\varrho} \sum_{y_n=-\infty}^{\infty} (1+\|y\|^p)\,\mathrm{e}^{-\|y\|/\varrho} \\
&\leq \|f\|_{p,\varrho}\, 2\, C_{p,\varrho}(1+\|x\|^p)\,\mathrm{e}^{-\|x\|/\varrho},
\end{aligned} \tag{A.10}
$$

which implies the bound $\|\bar{v}\| \leq 2C_{p,\varrho}\|f\|_{p,\varrho}$. Then, by the induction hypothesis, we have the bound on the norm of $\bar{g}$ as

$$\|\bar{g}\| \leq D_{p,\varrho}\|f\|_{p,\varrho} \tag{A.11}$$

for a constant $D_{p,\varrho}$ independent of L. As to $\bar{u}$, its norm is bounded by the norm of f, and by the induction hypothesis, we have the bound on the norm of $\bar{e}$

$$\|\bar{e}\| \leq E_{p,\varrho}\|f\|_{p,\varrho} \tag{A.12}$$

for a constant $E_{p,\varrho}$ independent of L. Combining these three bounds, we finally obtain the bound on the norm of g,

$$\|g\|_{p,\varrho} \leq \bar{C}_{p,\varrho}\,\|f\|_{p,\varrho}\,, \tag{A.13}$$

where $\bar{C}_{p,\varrho} = C_{p,\varrho} + D_{p,\varrho} + E_{p,\varrho}$ is a constant which may depend on p and ϱ, but not on the lattice size L. This completes the proof.

B. Relation between g and g_∞

We consider a closed form f which is a local composite field of the gauge field in the sense specified in section 3 with a reference point x_0. From its locality property, the norm of f is bounded by a constant C_1 independent of L for the fixed x_0, p and ρ,

$$\|f\|_{x_0,p,\rho} \leq C_1\,. \tag{B.1}$$

Then lemma 1 asserts that there exist two forms g and Δf which satisfies

$$f(x) = d^* g(x) + \Delta f(x), \tag{B.2}$$

$$\|g\|_{x_0,p,\rho} \leq C_2, \qquad |\Delta f_{\mu_1\ldots\mu_k}(x)| \leq C_3 L^p \mathrm{e}^{-L/2\varrho}. \tag{B.3}$$

On the other hand, f can be expressed as

$$f(x) = f_\infty(x) + \Delta f_\infty(x), \tag{B.4}$$

where f_∞ is a local form defined in the infinite lattice which satisfies

$$|f_{\mu_1\ldots\mu_k\infty}(x)| \leq c_1(1+\|x-x_0\|^p)\,\mathrm{e}^{-\|x-x_0\|/\varrho}\,, \tag{B.5}$$

and Δf_∞ is a finite volume correction which satisfies the bound

$$|\Delta f_{\mu_1\ldots\mu_k\infty}(x)| \leq c_2\, L^p\, \mathrm{e}^{-L/\varrho}\,. \tag{B.6}$$

If f_∞ is also a closed form,

$$d^* f_\infty(x) = 0\,, \tag{B.7}$$

then the original Poincaré lemma [29] asserts that there exists a form g_∞ such that

$$f_\infty(x) = d^* g_\infty(x). \tag{B.8}$$

Then the question we address in this appendix is the relation between g and g_∞ restricted with the periodic gauge fields. We can show that the difference between these two forms is a finite-volume correction which is suppressed exponentially in L. Namely, we have

Lemma 9.

$$g(x) = g_\infty(x) + \Delta g(x) \qquad (x \in \Gamma_{n[x_0]}) \tag{B.9}$$

$$|\Delta g(x)| \leq c_3 L^p \mathrm{e}^{-L/2\rho}\,. \tag{B.10}$$

Proof. The proof of lemmas 1, 2 holds true even if the lattice size is taken to be infinity $L \to \infty$, because of the locality property of the class of forms f in consideration, and it actually reduces in this limit to the proof of the original Poincaré lemma in [29]. Then the constructed g simply reduces to g_∞. Therefore, it suffices to compare these two solutions at a finite L and in the limit $L \to \infty$ in each step of the induction of the proof of the lemma.

For the case $n = 1$, 0-form f can be expressed as an exterior differential of g which is defined by

$$g(x) = \sum_{y_1 = x_{01} - L/2}^{x_1} f(y)\, dx_1\,. \tag{B.11}$$

But we can rewrite it as

$$\begin{aligned} g(x) &= \sum_{y_1 = x_{01} - L/2}^{x_1} \left(f_\infty(y) + \Delta f_\infty(y)\right) dx_1 \\ &= g_\infty(x) - \sum_{y_1 = -\infty}^{x_{01} - L/2 - 1} f_\infty(y)\ dx_1 + \sum_{y_1 = x_{01} - L/2}^{x_1} \Delta f_\infty(y)\, dx_1 \\ &= g_\infty(x) + \Delta g(x)\,, \end{aligned} \tag{B.12}$$

where $|\Delta g_1(x)| \leq \kappa L^p e^{-L/2\varrho}$. The bound on Δg follows from the locality property of f_∞ and Δf_∞. When f is 1-form, $g = g_\infty = 0$. Thus the statement is proved for $n = 1$.

We next consider the case $n > 1$. In this case, for a closed form $f = u dx_n + v$, g is constructed from v and two closed forms in $n - 1$ dimensions defined by

$$\begin{aligned} \bar{v}(x) &\equiv \sum_{y_n = -L/2}^{L/2-1} v(y)\,, \qquad (x_1, \ldots, x_{n-1}, y_n)\,, \\ \bar{u}(x) &\equiv u(x)|_{x_n = x_{0n} + L/2 - 1}\,. \end{aligned} \tag{B.13}$$

$\bar{v}$ and $\bar{u}$ lead to the forms $\bar{g}$ and $\bar{e}$, respectively, as

$$\begin{aligned} \bar{v} &= \bar{d}^* \bar{g} + \Delta \bar{v}\,, \\ \bar{u} &= \bar{d}^* \bar{e} + \Delta \bar{u}\,. \end{aligned} \tag{B.14}$$

Then g is given by

$$g(x) = \delta_{x_n,x_{0n}}\bar{g}(x) + h(x) + \bar{e}(x)\,dx_n\,, \tag{B.15}$$

where

$$h(x) \equiv (-1)^k \sum_{y_n=x_{0n}-L/2}^{x_n} \{v(y) - \delta_{y_n,x_{0n}}\bar{v}\}dx_n\,. \tag{B.16}$$

Since it holds true that

$$\begin{aligned}
\bar{v}(x) &= \sum_{y_n=-L/2}^{L/2-1} \{v_\infty(y) + \Delta v_\infty(y)\} \\
&= \bar{v}_\infty(x) - \sum_{y_n=-\infty}^{-L/2-1} v_\infty(y) - \sum_{y_n=L/2}^{\infty} v_\infty(y) + \sum_{y_n=-L/2}^{L/2-1} \Delta v_\infty(y)\,, \\
\bar{u}(x) &= \{u_\infty(x) + \Delta u_\infty(x)\}|_{x_n=x_{0n}+L/2-1},
\end{aligned} \tag{B.17}$$

we have

$$\begin{aligned}
\bar{v}(x) &= \bar{v}_\infty(x) + \Delta\bar{v}(x)\,; \qquad |\Delta\bar{v}_{\mu_1\ldots\mu_k}(x)| \le \kappa_1 L^p e^{-L/2\varrho}, \\
\bar{u}(x) &= \bar{u}_\infty(x) + \Delta\bar{u}(x)\,; \qquad \bar{u}_\infty(x) = 0, \quad |\Delta\bar{u}_{\mu_1\ldots\mu_{k-1}}(x)| \le \kappa_2 L^p e^{-L/2\varrho}\,.
\end{aligned} \tag{B.18}$$

Then, by the induction hypothesis, we can infer that

$$\begin{aligned}
\bar{g}(x) &= \bar{g}_\infty(x) + \Delta\bar{g}(x)\,; \qquad |\Delta\bar{g}_{\mu_1\ldots\mu_{k+1}}(x)| \le \kappa_1' L^p e^{-L/2\varrho}\,, \\
\bar{e}(x) &= \bar{e}_\infty(x) + \Delta\bar{e}(x)\,; \qquad \bar{e}_\infty(x) = 0\,, \quad |\Delta\bar{e}_{\mu_1\ldots\mu_k}(x)| \le \kappa_2' L^p e^{-L/2\varrho}\,.
\end{aligned} \tag{B.19}$$

Also we have

$$\begin{aligned}
h(x) &= (-1)^k \sum_{y_n=x_{0n}-L/2}^{x_n} \{v_\infty(y) - \delta_{y_n,x_{0n}}\bar{v}_\infty\}dx_n + \\
&\quad +(-1)^k \sum_{y_n=x_{0n}-L/2}^{x_n} \{\Delta v_\infty(y) - \delta_{y_n,x_{0n}}\Delta\bar{v}_\infty\}dx_n \\
&= h_\infty(x) - (-1)^k \sum_{y_n=-\infty}^{x_{0n}-L/2-1} \{v_\infty(y) - \delta_{y_n,x_{0n}}\bar{v}_\infty\}dx_n + \\
&\quad +(-1)^k \sum_{y_n=x_{0n}-L/2}^{x_n} \{\Delta v_\infty(y) - \delta_{y_n,x_{0n}}\Delta\bar{v}_\infty\}dx_n \\
&= h_\infty(x) + \Delta h(x)\,; \qquad |\Delta h_{\mu_1\ldots\mu_{k+1}}(x)| \le \kappa_3 L^p e^{-L/2\varrho}\,.
\end{aligned} \tag{B.20}$$

Then we can infer

$$g(x) = g_\infty(x) + \Delta g(x)\,, \tag{B.21}$$

where

$$\Delta g(x) = \delta_{x_n,x_{0n}}\Delta\bar{g}(x) + \Delta h(x) + \Delta\bar{e}(x)\,dx_n\,; \qquad |\Delta g_{\mu_1\ldots\mu_{k+1}}(x)| \le \kappa_4 L^p \mathrm{e}^{-L/2\rho}\,. \tag{B.22}$$

This result gives the proof of our statement.

C. A proof of the bounds on $q_{[m]}(x)$, $\phi_{[m]\mu\nu}(x)$ and $\gamma_{[m,w]}$

The first bound follows from the result eqs. (6.72) and (6.73), which is obtained through the cohomological analysis in the infinite lattice: for $U(x,\mu) = V_{[m]}(x,\mu)$, we have

$$\begin{aligned} q_{[m]}(x) &= q_{\infty}(x)|_{U=V_{[m]}} + \Delta q(x)|_{U=V_{[m]}} \\ &= \alpha + \beta_{\mu\nu}\frac{2\pi m_{\mu\nu}}{L^2} + \gamma\epsilon_{\mu\nu\rho\sigma}\frac{(2\pi)^2 m_{\mu\nu}m_{\rho\sigma}}{L^4} + \\ &\quad + \partial_\mu^*\left\{k_{\mu\infty}(x)|_{U=V_{[m]}} + \Delta k_{\mu\infty}(x)|_{U=V_{[m]}}\right\}. \end{aligned} \tag{C.1}$$

In the second expression, the gauge-invariant local current $k_{\mu\infty}(x)|_{U=V_{[m]}}$ should depend on the gauge field through its field tensor which is now constant for $V_{[m]}(x,\mu)$. Then by the translational invariance the current should not depend on the site x and its divergence vanishes identically. Then the bound follows immediately.

In order to show the second bound, we first recall that $\phi_{\mu\nu}(x)$ is calculated form the current $j_\mu(x,y)$ which is obtained by the first-derivative of the topological field $q(x)$ with respect to the vector potential $\tilde{A}_\mu(x)$. This current is evaluated for $V_{[m]}(x,\mu)$ as

$$\begin{aligned} j_\nu(x,y)|_{V_{[m]}} &= \int_0^1 dt \left(\frac{\partial q(x)}{\partial \tilde{A}_\nu(y)}\right)_{\tilde{A}\to t\tilde{A}}\Bigg|_{\tilde{A}=0} \\ &= \left(\frac{\partial q_\infty(x)}{\partial \tilde{A}_\nu(y)}\right)_{\tilde{A}=0} + \left(\frac{\partial \Delta q_\infty(x)}{\partial \tilde{A}_\nu(y)}\right)_{\tilde{A}=0} \\ &= \sum_{n\in\mathbb{Z}^4}\left(\frac{\partial q_\infty(x)}{\partial A_\nu(y+nL)}\right)_{U=V_{[m]}} + \left(\frac{\partial \Delta q_\infty(x)}{\partial \tilde{A}_\nu(y)}\right)_{\tilde{A}=0} \qquad (x,y\in\Gamma_4)\,, \end{aligned} \tag{C.2}$$

where in the third line we have used eq. (6.67) and in the last line we have taken into account that $\tilde{A}_\mu(x)$ is a periodic field in the infinite lattice. $A_\mu(x)$ is the vector potential introduced in [29].

The term with $n=0$ in the first term of the r.h.s.,

$$\left(\frac{\partial q_\infty(x)}{\partial A_\nu(y)}\right) \qquad (x,y\in\mathbb{Z}^4)\,, \tag{C.3}$$

is now defined on the infinite lattice. This term is related to the corresponding current $j_{\mu\infty}(x,y)$ in the infinite lattice which appears in the original cohomological analysis through the parameter integral,

$$j_{\nu\infty}(x,y) = \int_0^1 dt\left(\frac{\partial q_\infty(x)}{\partial A_\nu(y)}\right)_{A\to tA}. \tag{C.4}$$

From this current, the tensor field $\phi_{\mu\nu\infty}(x,y)$ descends as

$$j_{\nu\infty}(x,y) = \theta_{\nu\mu\infty}(x,y)\overleftarrow{\partial_\mu^*}\,, \qquad \frac{1}{2}\sum_{z\in\mathbb{Z}_4}\theta_{\mu\nu\infty}(z,x) = \phi_{\mu\nu\infty}(x)\,, \tag{C.5}$$

and it is evaluated further as

$$\phi_{\mu\nu\infty}(x) = \beta_{\mu\nu} + \gamma\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}(x) + \partial_\lambda^* t_{\lambda\mu\nu\infty}(x)\,. \tag{C.6}$$

But we note here that in the above analysis we might have applied the Poincaré lemma before doing the parameter integral. Namely, we have

$$\left(\frac{\partial q_\infty(x)}{\partial A_\nu(y)}\right) = \check{\theta}_{\nu\mu\infty}(x,y)\overleftarrow{\partial}^*_\mu\,, \qquad \frac{1}{2}\sum_{z\in\mathbb{Z}_4}\check{\theta}_{\mu\nu\infty}(z,x) = \check{\phi}_{\mu\nu\infty}(x) \tag{C.7}$$

and

$$\check{\phi}_{\mu\nu\infty}(x) = \check{\beta}_{\mu\nu} + \check{\gamma}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}(x) + \partial^*_\lambda \check{t}_{\lambda\mu\nu\infty}(x)\,. \tag{C.8}$$

Since the parameter integral of this expression should reproduce the above result eq. (C.6), $\check{\beta}_{\mu\nu}$, $\check{\gamma}$ and $\check{t}_{\lambda\mu\nu\infty}(x)$ are related to $\beta_{\mu\nu}$, γ and $t_{\lambda\mu\nu\infty}(x)$, respectively as follows:

$$\check{\beta}_{\mu\nu} = \beta_{\mu\nu}\,, \qquad \frac{1}{2}\check{\gamma} = \gamma\,, \qquad \int_0^1 dt\;\check{t}_{\lambda\mu\nu}(x)\big|_{A\to tA} = t_{\lambda\mu\nu}(x)\,. \tag{C.9}$$

The tensor field $\phi_{\mu\nu}(x)$ on the finite lattice, on the other hand, is obtained from $\theta_{\mu\nu}(x,y)$ as in eqs. (6.16), (6.19) and (6.20). The difference between the solutions of these two lemma is, as shown in the appendix B, an exponentially small correction and therefore we can infer

$$\theta_{\mu\nu}(x,y)|_{V_{[m]}} = \check{\theta}_{\mu\nu\infty}(x,y)|_{V_{[m]}} + \Delta\theta_{\mu\nu}(x,y)\,, \qquad |\Delta\theta_{\mu\nu}(x,y)| \le c_1 L^{\sigma_1}\mathrm{e}^{-L/2\rho}\,. \tag{C.10}$$

This immediately implies

$$\phi_{[m]\mu\nu}(x) = \check{\phi}_{\mu\nu\infty}(x)|_{V_{[m]}} + \Delta\phi_{\mu\nu}(x)|_{V_{[m]}}\,, \qquad |\Delta\phi_{\mu\nu}(x)| \le c_2 L^{\sigma_1}\mathrm{e}^{-L/2\rho}\,, \tag{C.11}$$

while $\check{\phi}_{\nu\mu\infty}(x)$ evaluates for $V_{[m]}(x,\mu)$ as

$$\check{\phi}_{\mu\nu\infty}(x)|_{V_{[m]}} = \beta_{\mu\nu} + 2\gamma\epsilon_{\mu\nu\rho\sigma}\frac{2\pi m_{\rho\sigma}}{L^2}\,, \tag{C.12}$$

because the gauge-invariant tensor field $\check{t}_{\lambda\mu\nu\infty}(x)$ is a constant for this case. This proves the second bound.

As to the third bound, we first recall the fact that the dependence of $\gamma_{[m,w]}$ on the gauge potential $\tilde{A}_\mu(x)$ is almost excluded except the dependence on the Wilson line w_μ. This dependence on $\tilde{A}_\mu(x)$ (or w_μ) is in fact exponentially small. One way to see this is to repeat the cohomological analysis for $\epsilon_{\rho\sigma\tau\lambda}\,\gamma_{[m,w]}(x) = (1/2)\sum_{z\in\Gamma_4}\omega_{\rho\sigma\tau\lambda}(z,x)$ which satisfies $\partial^*_\rho\epsilon_{\rho\sigma\tau\lambda}\,\gamma_{[m,w]}(x) = 0$, regarding it as a tensor field of rank four, just like for $\phi_{\mu\nu}(x)$. Then we obtain an expression similar to eq. (6.59). But in this case the tensor fields which correspond to ξ and $\Delta\Xi$ in eq. (6.59) should be antisymmetric tensors of rank six and should vanish identically in four dimensions. Therefore we have

$$\epsilon_{\rho\sigma\tau\lambda}\gamma_{[m,w]} = \epsilon_{\rho\sigma\tau\lambda}\gamma_{[m,w]}|_{\tilde{A}=0} + \sum_{y\in\Gamma_4}\Delta j_{\rho\sigma\tau\lambda\mu}(x,y)\tilde{A}_\mu(y)\,, \qquad |\Delta j_{\rho\sigma\tau\lambda\mu}(x,y)| \le c_3 L^{\sigma_3}\mathrm{e}^{-L/2\rho} \tag{C.13}$$

and the difference $\gamma_{[m,w]} - \gamma_{[m,w]}|_{\tilde{A}=0}$ is indeed exponentially small.

We next recall that $\gamma_{[m,w]}$ is calculated form the current $j_{\mu\nu\rho}(x,y)$ which is obtained by the first-derivative of the tensor field $\phi_{\mu\nu}(x)$ with respect to the vector potential $\tilde{A}_\mu(x)$. This current is evaluated for $V_{[m]}(x,\mu)$ as

$$\begin{aligned} j_{\mu\nu\rho}(x,y)|_{V_{[m]}} &= \int_0^1 dt \left(\frac{\partial\phi_{\mu\nu}(x)}{\partial\tilde{A}_\rho(y)}\right)_{\tilde{A}\to t\tilde{A}}\Bigg|_{\tilde{A}=0} \\ &= \left(\frac{\partial\phi_{\mu\nu}(x)}{\partial\tilde{A}_\rho(y)}\right)_{\tilde{A}=0} . \end{aligned} \tag{C.14}$$

The tensor field $(\partial\phi_{\mu\nu}(x)/\partial\tilde{A}_\rho(y))_{\tilde{A}=0}$ in turn is calculated from the derivative of $j_\mu(x,z)$ with respect to the vector potential, that is the second derivative of the topological field $q(x)$:

$$\begin{aligned} \left(\frac{\partial j_\mu(x,z)}{\partial\tilde{A}_\rho(y)}\right)_{\tilde{A}=0} &= \int_0^1 dt\, t \left(\frac{\partial^2 q(x)}{\partial\tilde{A}_\rho(y)\partial\tilde{A}_\mu(z)}\right)_{\tilde{A}\to\tilde{t}A}\Bigg|_{\tilde{A}=0} \\ &= \frac{1}{2}\left(\frac{\partial^2 q(x)}{\partial\tilde{A}_\rho(y)\partial\tilde{A}_\mu(z)}\right)_{\tilde{A}=0} \\ &= \sum_{m,n\in\mathbb{Z}^4} \frac{1}{2}\left(\frac{\partial^2 q_\infty(x)}{\partial A_\rho(y+mL)\partial A_\mu(z+nL)}\right)_{U=V_{[m]}} + \frac{1}{2}\left(\frac{\partial^2 \Delta q_\infty(x)}{\partial\tilde{A}_\rho(y)\partial\tilde{A}_\mu(z)}\right)_{\tilde{A}=0}, \\ &\qquad (x,y,z\in\Gamma_4)\,. \end{aligned} \tag{C.15}$$

The term with $m=n=0$ in the r.h.s. of eq. (C.15)

$$\left(\frac{\partial^2 q_\infty(x)}{\partial A_\rho(y)\partial A_\mu(z)}\right) \tag{C.16}$$

is again defined in the infinite lattice. It is first related to the differentiation of current $j_{\mu\infty}(x,y)$ with respect to the vector potential, by the parameter integral:

$$\left(\frac{\partial j_{\mu\infty}(x,z)}{\partial A_\rho(y)}\right) = \int_0^1 dt\, t \left(\frac{\partial^2 q_\infty(x)}{\partial A_\rho(y)\partial A_\mu(z)}\right)_{A\to tA} . \tag{C.17}$$

From this current, $(\partial\phi_{\mu\nu\infty}(x)/\partial A_\rho(y))$ is then obtained by the applications of the Poincaré and this in turn is related to the current $j_{\mu\nu\rho\infty}(x,y)$ in the infinite lattice as

$$j_{\mu\nu\rho\infty}(x,y) = \int_0^1 dt \left(\frac{\partial\phi_{\mu\nu\infty}(x)}{\partial A_\rho(y)}\right)_{A\to tA} . \tag{C.18}$$

$j_{\mu\nu\rho\infty}(x,y)$ is evaluated by the cohomological analysis in the infinite lattice as

$$j_{\mu\nu\rho\infty}(x,y) = \xi_{\mu\nu\rho\sigma\infty}(x,y)\overleftarrow{\partial}^*_\sigma, \tag{C.19}$$

$$\xi_{\mu\nu\rho\sigma\infty}(x,y) = 2\gamma\,\epsilon_{\mu\nu\rho\sigma}\delta_{x,y-\hat{\mu}-\hat{\nu}} + \partial^*_\lambda\kappa_{\lambda\mu\nu\rho\sigma\infty}(x,y) + \theta_{\mu\nu\rho\sigma\tau\infty}(x,y)\overleftarrow{\partial}^*_\tau\,. \tag{C.20}$$

But we note again that the same cohomological analysis may be applied to the current $j'_{\mu\nu\rho\infty}(x,y)$ which is obtained starting from eq. (C.16), but without the parameter integrals.

The result reads

$$j'_{\mu\nu\rho\infty}(x,y) = \xi'_{\mu\nu\rho\sigma\infty}(x,y)\overleftarrow{\partial}^*_\sigma\,, \tag{C.21}$$

$$\xi'_{\mu\nu\rho\sigma\infty}(x,y) = 2\gamma'\,\epsilon_{\mu\nu\rho\sigma}\delta_{x,y-\hat{\mu}-\hat{\nu}} + \partial^*_\lambda\kappa'_{\lambda\mu\nu\rho\sigma\infty}(x,y) + \theta'_{\mu\nu\rho\sigma\tau\infty}(x,y)\overleftarrow{\partial}^*_\tau\,, \tag{C.22}$$

where γ', κ' and θ' are related to their counterparts, γ, κ and θ, as

$$\gamma'\int_0^1 ds\int_0^1 dt\,t = \frac{1}{2}\gamma' = \gamma\,, \tag{C.23}$$

$$\int_0^1 ds\int_0^1 dt\,t\,\kappa'_{\lambda\mu\nu\rho\sigma\infty}(x,y)|_{A\to stA} = \kappa_{\lambda\mu\nu\rho\sigma\infty}(x,y), \tag{C.24}$$

$$\int_0^1 ds\int_0^1 dt\,t\,\theta'_{\mu\nu\rho\sigma\tau\infty}(x,y)|_{A\to stA} = \theta_{\mu\nu\rho\sigma\tau\infty}(x,y)\,. \tag{C.25}$$

On the other hand, in obtaining $\gamma_{[m,w]}|_{\tilde{A}=0}$ from eqs. (C.14) and (C.15), by the repeat applications of the Poincaré lemma, we may consider to apply the original Poincaré lemma in the infinite lattice to the term with $m = n = 0$ in the r.h.s. of eq. (C.15), which is defined in the infinite lattice, and also to its descendants. The difference between the solutions of these two lemma is, as shown in the appendix B, an exponentially small correction and then we can infer

$$\gamma_{[m,w]}|_{\tilde{A}=0} = \frac{1}{2}\gamma' + \Delta\gamma\,; \qquad |\Delta\gamma| \le c_4 L^{\sigma_2}\mathrm{e}^{-L/2\rho}\,. \tag{C.26}$$

Taking into account that $\gamma' = 2\gamma$, we obtain the third bound.

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